CS 294-73
Software Engineering for Scientific Computing

Lecture 10:Dense Linear Algebra

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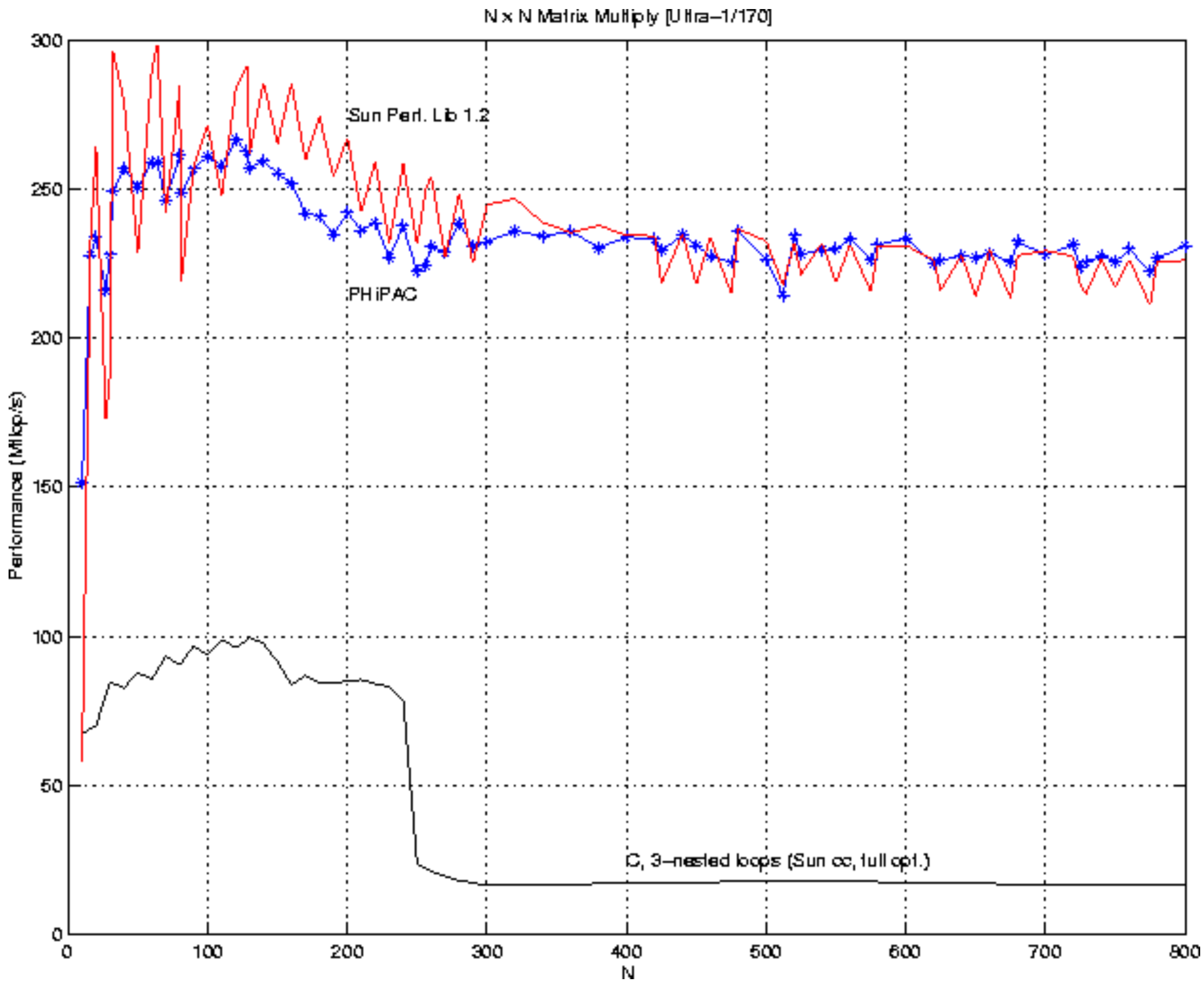
Outline

- What is Dense Linear Algebra?
- Where does the time go in an algorithm?
  - Moving Data in Memory hierarchies
- Case studies
  - Matrix Multiplication
  - Gaussian Elimination

What is dense linear algebra?

- Not just Basic Linear Algebra Subroutines (eg matmul)
- Linear Systems: Ax=b
- Least Squares: choose x to minimize $||Ax-b||_2$
  - Overdetermined or underdetermined
  - Unconstrained, constrained, weighted
- Eigenvalues and vectors of Symmetric Matrices
  - Standard ($Ax = \lambda x$), Generalized ($Ax=\lambda Bx$)
- Eigenvalues and vectors of Unsymmetric matrices
  - Eigenvalues, Schur form, eigenvectors, invariant subspaces
  - Standard, Generalized
- Singular Values and vectors (SVD)
  - Standard, Generalized
- Different matrix structures
  - Real, complex; Symmetric, Hermitian, positive definite; dense, triangular, banded …
- Level of detail
  - Simple Driver
  - Expert Drivers with error bounds, extra-precision, other options
  - Lower level routines ("apply certain kind of orthogonal transformation",…)

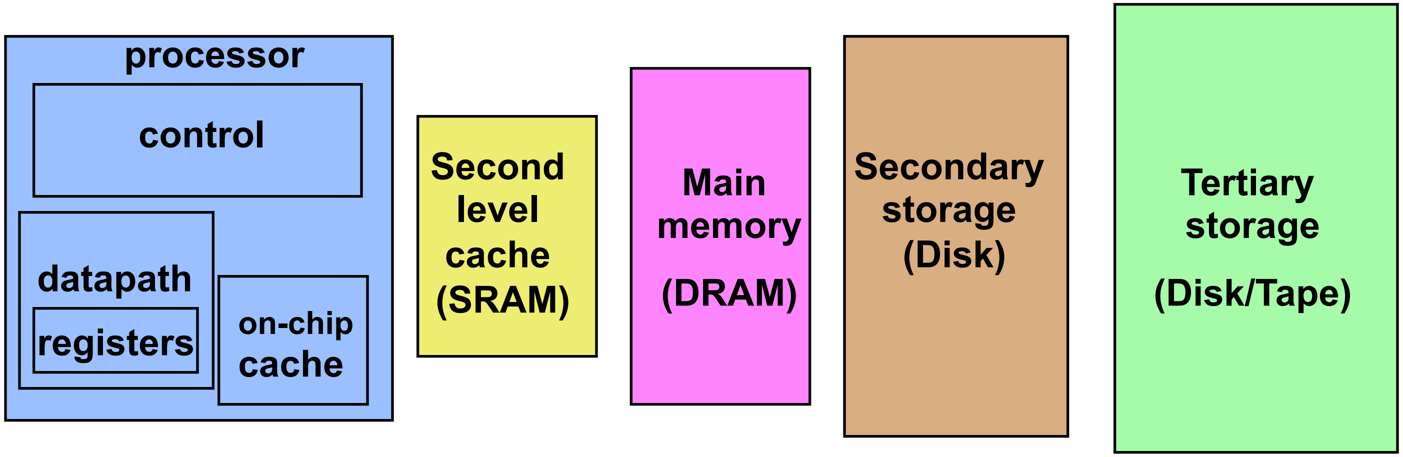
Matrix-multiply, optimized several ways



Speed of n-by-n matrix multiply on Sun Ultra-1/170, peak = 330 MFlops

Where does the time go?

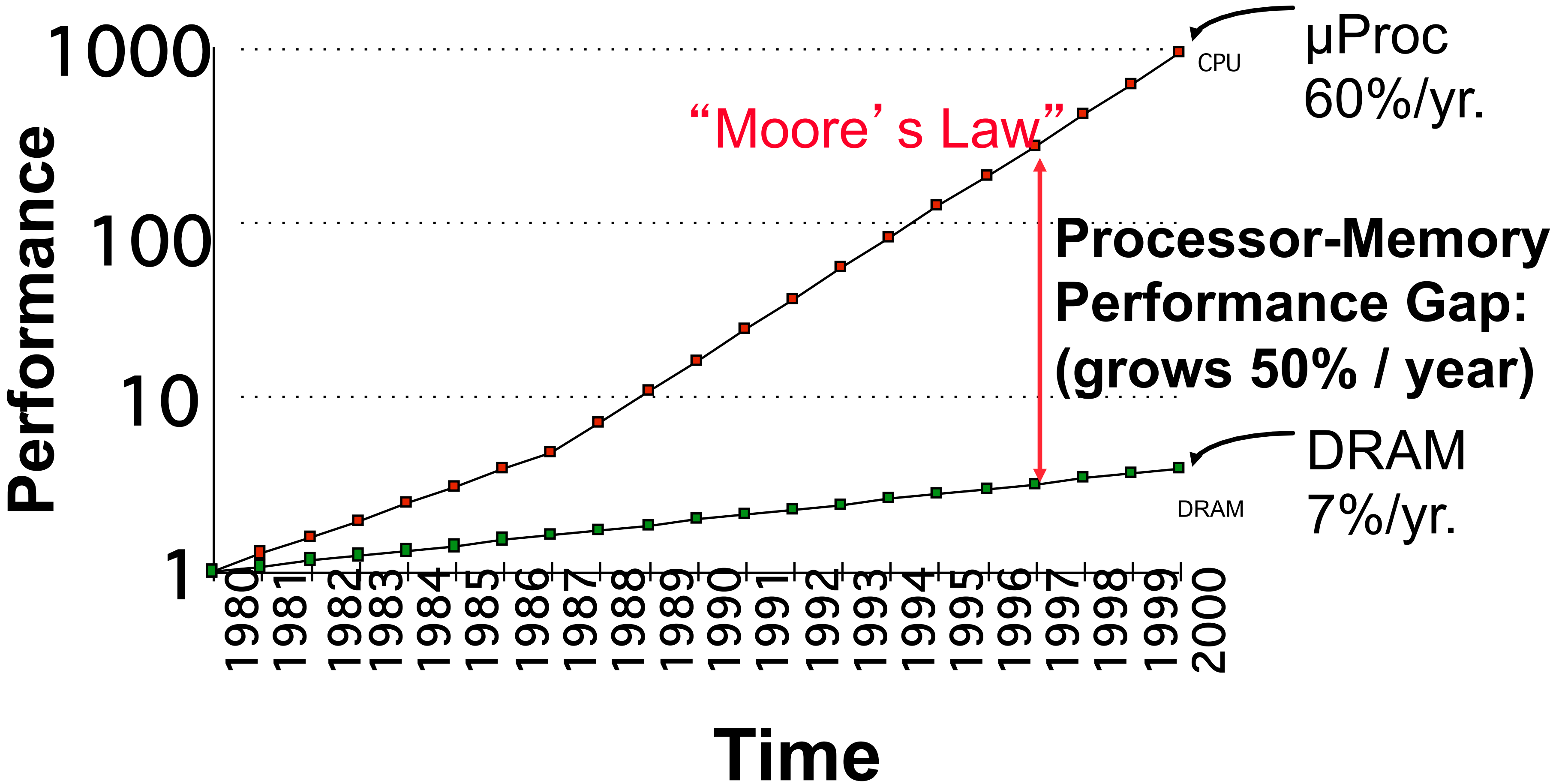
- Hardware organized as Memory Hierarchy
  - Try to keep frequently accessed data in fast, but small memory
  - Keep less frequently accessed data in slower, but larger memory
- Time(flops) ≅ Time(on-chip cache access) << Time(slower mem access)
  - Need algorithms that minimize accesses to slow memory, i.e. "minimize communication"



Speed	1ns	10ns	100ns	10ms	10sec
Size	KB	MB	GB	TB	PB

Minimizing communication gets more important every year

- Memory hierarchies are getting deeper
    - Processors get faster more quickly than memory



Using a Simple Model of Memory to Optimize

- Assume just 2 levels in the hierarchy, fast and slow
- All data initially in slow memory
  - m = number of memory elements (words) moved between fast and slow memory
  - t_m = time per slow memory operation
  - f = number of arithmetic operations
  - t_f = time per arithmetic operation << t_m
  - $q = f / m$ average number of flops per slow memory access
- Minimum possible time = $f^* \, t_f$ when all data in fast memory
- Actual time
  - $f * t_f + m * t_m = f * t_f * (1 + \boxed{t_m/t_f} * 1/q)$
- Larger q means time closer to minimum $f * t_f$
  - $q \geq t_m/t_f$ needed to get at least half of peak speed

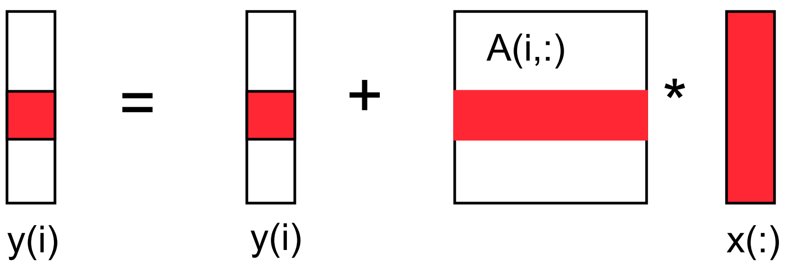
    ≥1000

Computational Intensity: **Key to algorithm efficiency**

Machine Balance: **Key to machine efficiency**

Warm up: Matrix-vector multiplication

```
{implements y = y + A*x}
for i = 1:n
        for j = 1:n
                y(i) = y(i) + A(i,j)*x(j)
```



Warm up: Matrix-vector multiplication

```
{read x(1:n) into fast memory}
{read y(1:n) into fast memory}
for i = 1:n
      {read row i of A into fast memory}
      for j = 1:n
              y(i) = y(i) + A(i,j)*x(j)
{write y(1:n) back to slow memory}
```

- m = number of slow memory refs = $3n + n^2$
- f = number of arithmetic operations = $2n^2$
- $q = f / m \approx 2$

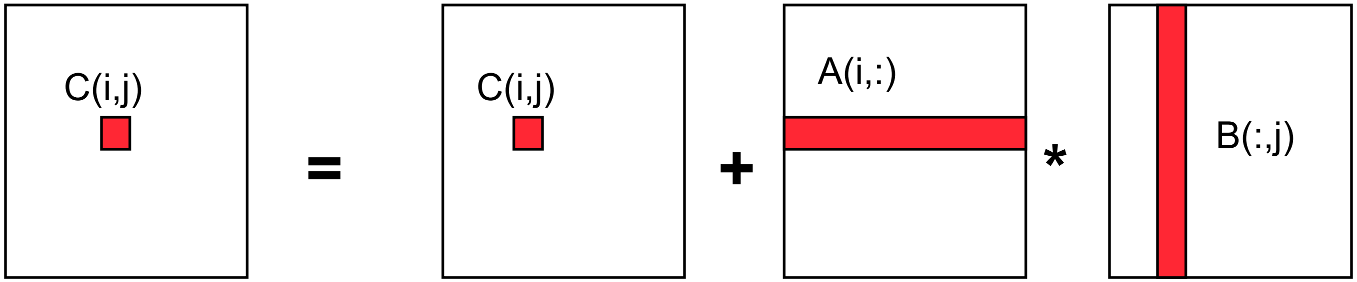
- Matrix-vector multiplication limited by slow memory speed

Naïve Matrix Multiply

```
{implements C = C + A*B}
for i = 1 to n
    for j = 1 to n
        for k = 1 to n
                C(i,j) = C(i,j) + A(i,k) * B(k,j)
```

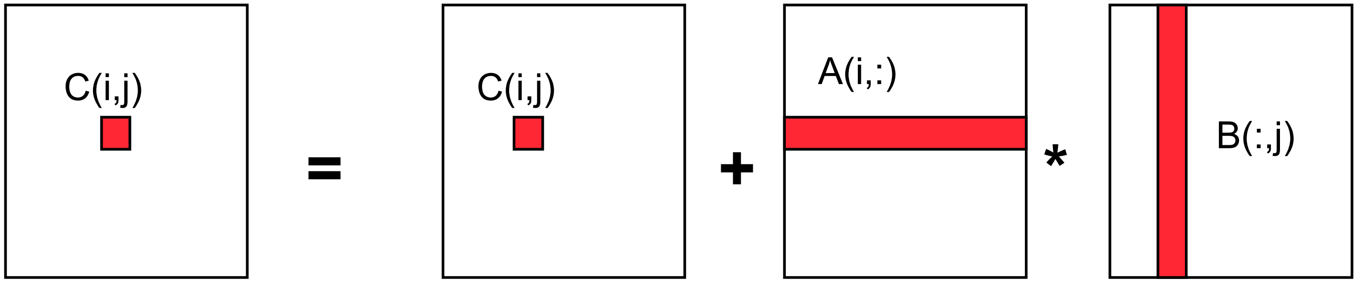
Algorithm has $2*n^3 = O(n^3)$ Flops and operates on $3*n^2$ words of memory

q potentially as large as $2*n^3 / 3*n^2 = O(n)$



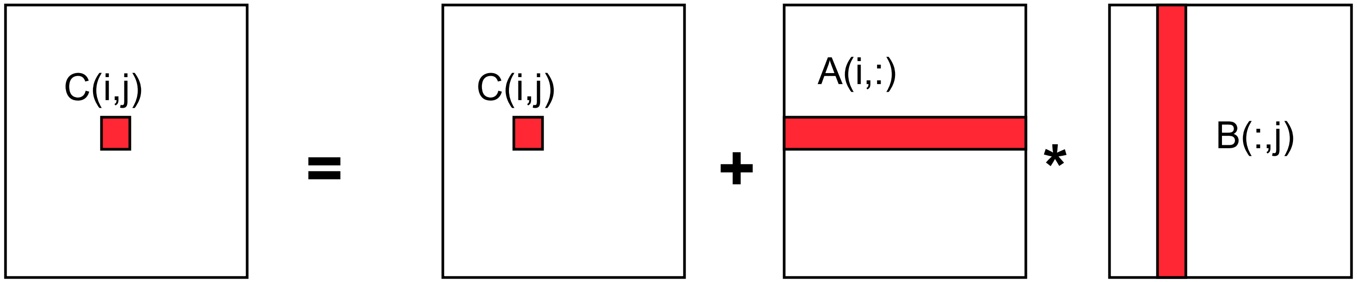
Naïve Matrix Multiply

```
{implements C = C + A*B}
for i = 1 to n
  {read row i of A into fast memory}
   for j = 1 to n
      {read C(i,j) into fast memory}
      {read column j of B into fast memory}
      for k = 1 to n
         C(i,j) = C(i,j) + A(i,k) * B(k,j)
      {write C(i,j) back to slow memory}
```



Naïve Matrix Multiply

```
{implements C = C + A*B}
for i = 1 to n
  {read row i of A into fast memory   …  n^2 total reads}
   for j = 1 to n
       {read C(i,j) into fast memory   …  n^2 total reads}
       {read column j of B into fast memory   …  n^3 total reads}
       for k = 1 to n
           C(i,j) = C(i,j) + A(i,k) * B(k,j)
       {write C(i,j) back to slow memory  …  n^2 total writes}
```



Naïve Matrix Multiply

Number of slow memory references on unblocked matrix multiply

$m = \quad n^3$ to read each column of B n times

$\quad + n^2$ to read each row of A once

$\quad + 2n^2$ to read and write each element of C once

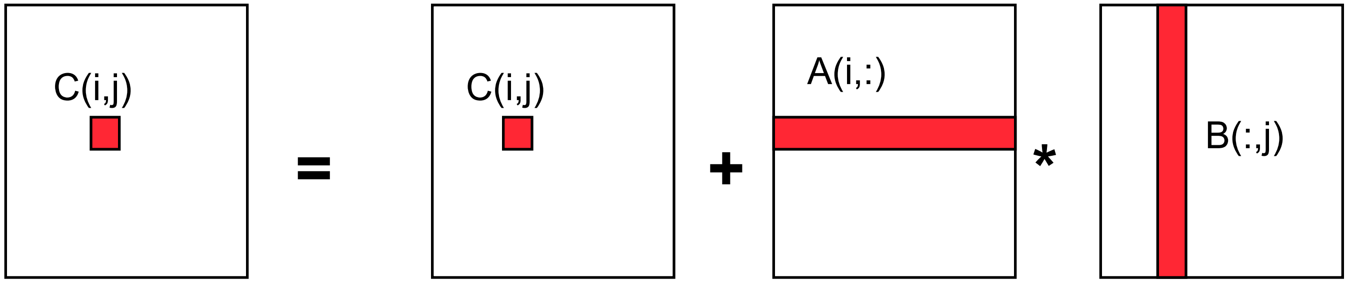
$\quad = n^3 + 3n^2$

So $q = f / m = 2n^3 / (n^3 + 3n^2)$

$\quad \approx 2$ for large n, no improvement over matrix-vector multiply

Inner two loops are just matrix-vector multiply, of row i of A times B

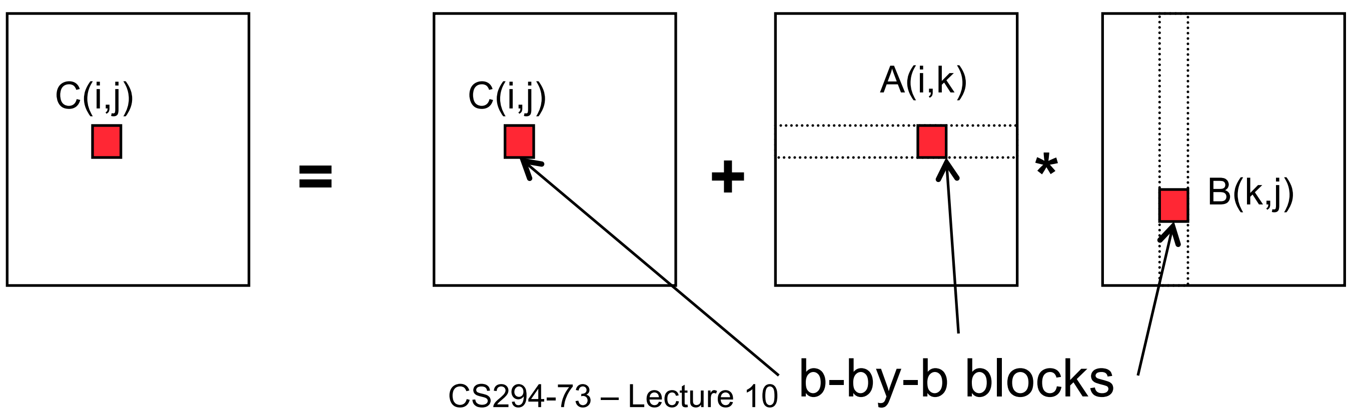
Similar for any other order of 3 loops



Blocked (Tiled) Matrix Multiply

Consider A,B,C to be (n/b)-by-(n/b) matrices of b-by-b blocks where b is called the block size; **assume fast memory holds 3 b-by-b blocks**

```
for i = 1 to n/b
  for j = 1 to n/b
    {read block C(i,j) into fast memory}
    for k = 1 to n/b
        {read block A(i,k) into fast memory}
        {read block B(k,j) into fast memory}
        C(i,j) = C(i,j) + A(i,k) * B(k,j)   {matrix multiply on b-by-b blocks}
    {write block C(i,j) back to slow memory}
```



Blocked (Tiled) Matrix Multiply

Recall:

m is amount memory traffic between slow and fast memory

matrix is n-by-n, arranged as (n/b)-by-(n/b) matrix of b-by-b blocks

$f = 2n^3 =$ number of floating point operations

$q = f / m =$ "computational intensity"

So:

$m = n^3/b$ read each block of B $(n/b)^3$ times, so $(n/b)^3 * b^2 = n^3/b$

$+ n^3/b$ read each block of A $(n/b)^3$ times

$+ 2n^2$ read and write each block of C once

$= 2n^3/b + 2n^2$

So computational intensity $q = f / m = 2n^3 / (2n^3/b + 2n^2)$

$\approx 2n^3 / (2n^3/b) = b$ for large n

So we can improve performance by increasing the blocksize b

Can be much faster than matrix-vector multiply (q=2)

Limits to Optimizing Matrix Multiply

- #slow_memory_references = m = $2n^3/b + 2n^2$
- Increasing b reduces m. How big can be we make b?
- Recall assumption that 3 b-by-b blocks C(i,j), A(i,k) and B(k,j) fit in fast memory, say of size M
  - Constraint: $3b^2 \le M$
- Tiled matrix multiply cannot reduce m below $\approx 2\ 3^{1/2}\ n^3 / M^{1/2}$

- Theorem (Hong & Kung, 1981): You can't do any better than this: Any reorganization of this algorithm (that uses only commutativity and associativity) is limited to #slow_memory_references = $\Omega(n^3 / M^{1/2})$ ($f(x) = \Omega(g(x))$ means that $|f(x)| \ge C\ |g(x)|$).

Limits to Optimizing All Dense Linear Algebra

- Theorem [Ballard, Demmel, Holtz, Schwartz, 2011]: Consider any algorithm that is "like 3 nested loops in matrix-multiply", running with fast memory size M. Then no matter how you optimize it, using only associativity and commutativity,

  $\#slow\_memory\_references = \Omega\ (\#flops / M^{1/2})$

- This applies to
  - Basic Linear Algebra Subroutines like matmul, triangular solve, …
  - Gaussian elimination, Cholesky, other variants …
  - Least squares
  - Eigenvalue problems and the SVD
  - Some operations on tensors, graph algorithms …
  - Some whole programs that call a sequence of these operations
  - Multiple levels of memory hierarchy, not just "fast and slow"
  - Dense and sparse matrices
    - #flops = $O(n^3)$ for dense matrices, usually much less for sparse
  - Sequential and parallel algorithms
    - Parallel case covered in CS267

Can we attain these lower bounds?

- Do algorithms in standard dense linear algebra libraries attain these bounds?
  - LAPACK (sequential), ScaLAPACK (parallel), versions offered by vendors
  - For some problems (eg matmul) they do, but mostly not
  - Note: these libraries are still fast, and should be your first choice on most problems!
- Are there other algorithms that do attain them?
  - Yes, some known for a while, some under development
- Two examples
  - Matmul (again, but for any memory hierarchy)
  - Gaussian Elimination

What if there are more than 2 levels of memory?

- Goal is to minimize communication between all levels
- The tiled algorithm requires finding a good block size
  - Machine dependent: b depends on fast memory size M
  - Need to "block" b x b matrix multiply in inner most loop
    - 1 level of memory ⇒ 3 nested loops (naïve algorithm)
    - 2 levels of memory ⇒ 6 nested loops
    - 3 levels of memory ⇒ 9 nested loops …

- Cache Oblivious Algorithms offer an alternative
  - Treat nxn matrix multiply as a set of smaller problems
  - Eventually, these will fit in cache
  - Will minimize # words moved between every level of memory hierarchy (between L1 and L2 cache, L2 and L3, L3 and main memory etc.) – at least asymptotically

Recursive Matrix Multiplication (RMM) (1/2)

- For simplicity: square matrices with $n = 2^m$
- $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = A \cdot B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

$$= \begin{pmatrix} A_{11}\cdot B_{11} + A_{12}\cdot B_{21} & A_{11}\cdot B_{12} + A_{12}\cdot B_{22} \\ A_{21}\cdot B_{11} + A_{22}\cdot B_{21} & A_{21}\cdot B_{12} + A_{22}\cdot B_{22} \end{pmatrix}$$

- True when each A_{ij} etc 1x1 or n/2 x n/2

```
func C = RMM (A, B, n)
   if n = 1, C = A * B, else
     {  C11 = RMM (A11 , B11 , n/2) + RMM (A12 , B21 , n/2)
        C12 = RMM (A11 , B12 , n/2) + RMM (A12 , B22 , n/2)
        C21 = RMM (A21 , B11 , n/2) + RMM (A22 , B21 , n/2)
        C22 = RMM (A21 , B12 , n/2) + RMM (A22 , B22 , n/2)  }
   return
```

Recursive Matrix Multiplication (2/2)

```
func C = RMM (A, B, n)
   if n=1, C = A * B, else
     {  C11 = RMM (A11 , B11 , n/2) + RMM (A12 , B21 , n/2)
        C12 = RMM (A11 , B12 , n/2) + RMM (A12 , B22 , n/2)
        C21 = RMM (A21 , B11 , n/2) + RMM (A22 , B21 , n/2)
        C22 = RMM (A21 , B12 , n/2) + RMM (A22 , B22 , n/2)  }
   return
```

A(n) = # arithmetic operations in RMM(. , . , n)
= $8 \cdot A(n/2) + 4(n/2)^2$ if $n > 1$, else 1
= $2n^3$ … same operations as usual, in different order

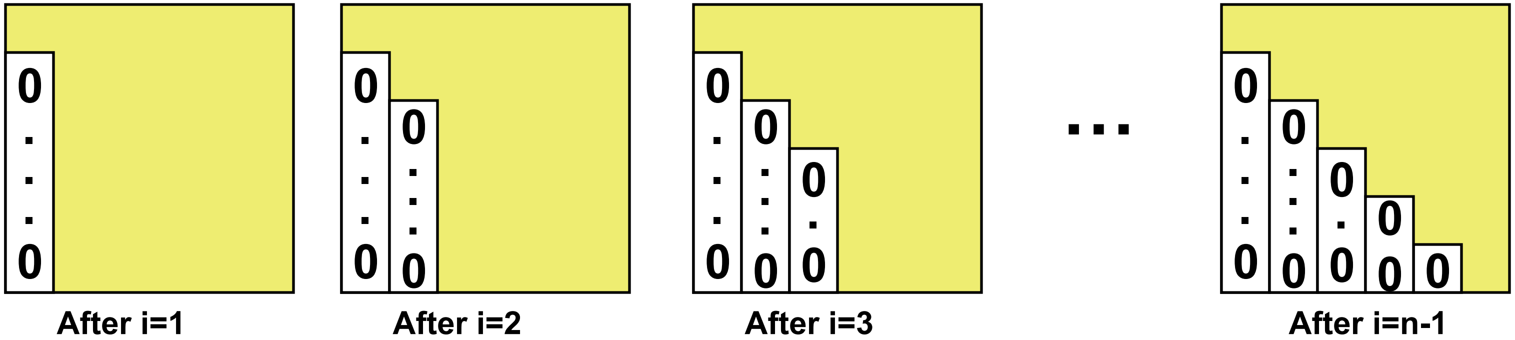
W(n) = # words moved between fast, slow memory by RMM(. , . , n)
= $8 \cdot W(n/2) + 4(n/2)^2$ if $3n^2 > M$, else $3n^2$
= $O(n^3 / (M)^{1/2} + n^2)$ … same as blocked matmul

Algorithm called "cache oblivious", because doesn't depend on M

Gaussian Elimination (GE) for solving Ax=b

- Add multiples of each row to later rows to make A upper triangular
- Solve resulting triangular system Ux = c by substitution

```
… for each column i
… zero it out below the diagonal by adding multiples of row i to later rows
for i = 1 to n-1
   … for each row j below row i
   for j = i+1 to n
       … add a multiple of row i to row j
       tmp = A(j,i);
       for k = i to n
           A(j,k) = A(j,k) - (tmp/A(i,i)) * A(i,k)
```



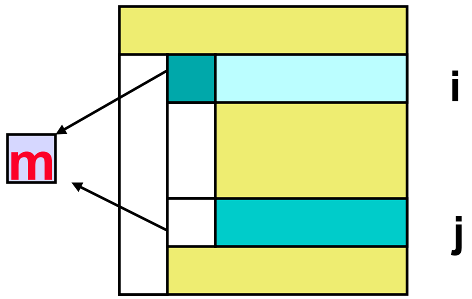
Refine GE Algorithm (1/5)

- Initial Version

```
… for each column i
… zero it out below the diagonal by adding multiples of row i to later rows
for i = 1 to n-1
   … for each row j below row i
   for j = i+1 to n
       … add a multiple of row i to row j
       tmp = A(j,i);
       for k = i to n
           A(j,k) = A(j,k) - (tmp/A(i,i)) * A(i,k)
```

- Remove computation of constant tmp/A(i,i) from inner loop.

```
for i = 1 to n-1
    for j = i+1 to n
        m = A(j,i)/A(i,i)
        for k = i to n
            A(j,k) = A(j,k) - m * A(i,k)
```



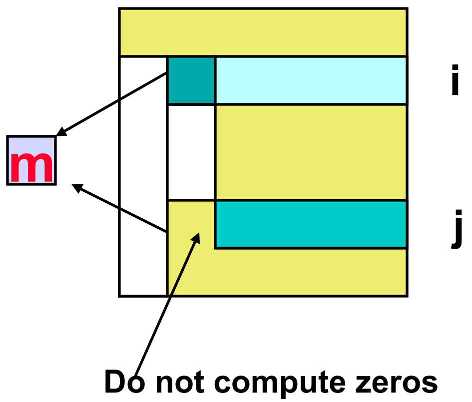
Refine GE Algorithm (2/5)

- Last version

```
for i = 1 to n-1
    for j = i+1 to n
        m = A(j,i)/A(i,i)
        for k = i to n
            A(j,k) = A(j,k) - m * A(i,k)
```

- Don't compute what we already know: zeros below diagonal in column i

```
for i = 1 to n-1
    for j = i+1 to n
        m = A(j,i)/A(i,i)
        for k = i+1 to n
            A(j,k) = A(j,k) - m * A(i,k)
```



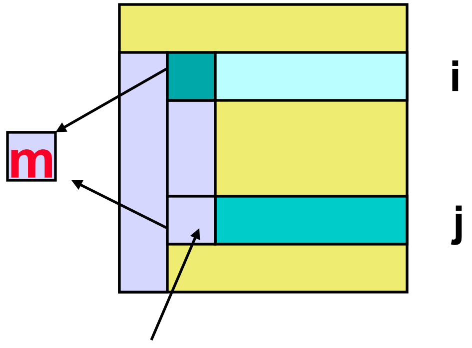
Refine GE Algorithm (3/5)

- Last version

```
for i = 1 to n-1
    for j = i+1 to n
        m = A(j,i)/A(i,i)
        for k = i+1 to n
            A(j,k) = A(j,k) - m * A(i,k)
```

- Store multipliers m below diagonal in zeroed entries for later use

```
for i = 1 to n-1
    for j = i+1 to n
        A(j,i) = A(j,i)/A(i,i)
        for k = i+1 to n
            A(j,k) = A(j,k) - A(j,i) * A(i,k)
```



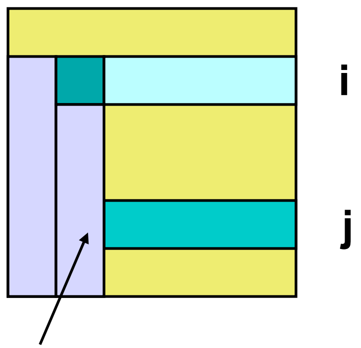
Refine GE Algorithm (4/5)

- Last version

```
for i = 1 to n-1
    for j = i+1 to n
        A(j,i) = A(j,i)/A(i,i)
        for k = i+1 to n
            A(j,k) = A(j,k) - A(j,i) * A(i,k)
```

- **Split Loop**

```
for i = 1 to n-1
    for j = i+1 to n
        A(j,i) = A(j,i)/A(i,i)
    for j = i+1 to n
        for k = i+1 to n
            A(j,k) = A(j,k) - A(j,i) * A(i,k)
```



Store all m's here before updating rest of matrix

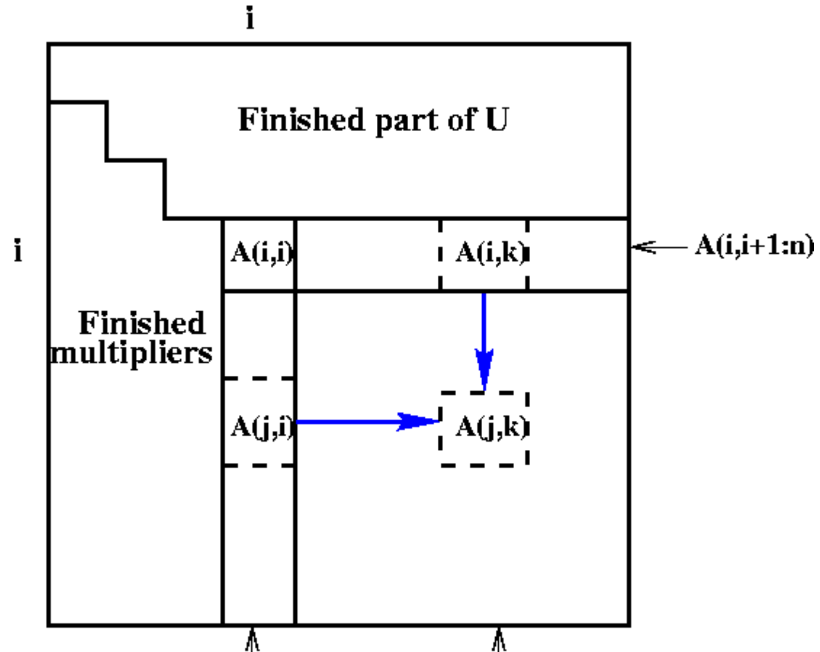
Refine GE Algorithm (5/5)

- Last version

```
for i = 1 to n-1
    for j = i+1 to n
        A(j,i) = A(j,i)/A(i,i)
    for j = i+1 to n
        for k = i+1 to n
            A(j,k) = A(j,k) - A(j,i) * A(i,k)
```

- Express using matrix operations (BLAS)

Work at step i of Gaussian Elimination




```
for i = 1 to n-1
    A(i+1:n,i) = A(i+1:n,i) * ( 1 / A(i,i) )
        … BLAS 1 (scale a vector)
    A(i+1:n,i+1:n) = A(i+1:n , i+1:n )
        - A(i+1:n , i) * A(i , i+1:n)
        … BLAS 2 (rank-1 update)
```

What GE really computes

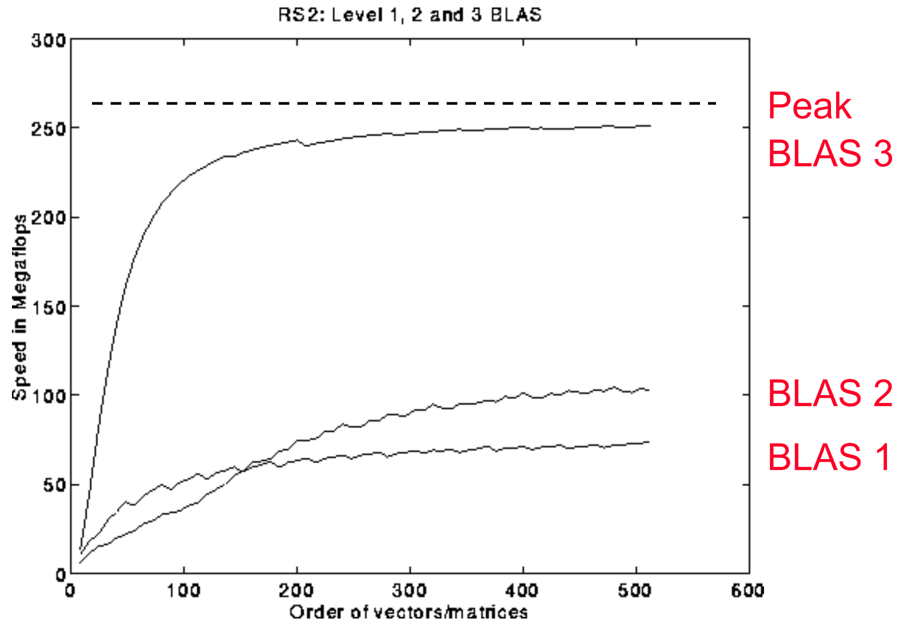
```
for i = 1 to n-1
    A(i+1:n,i) = A(i+1:n,i) / A(i,i)     … BLAS 1 (scale a vector)
    A(i+1:n,i+1:n) = A(i+1:n , i+1:n ) - A(i+1:n , i) * A(i , i+1:n)   … BLAS 2 (rank-1 update)
```

- Call the strictly lower triangular matrix of multipliers M, and let L = I+M
- Call the upper triangle of the final matrix U
- *Lemma (LU Factorization):* If the above algorithm terminates (does not divide by zero) then A = L*U
- Solving A*x=b using GE
  - Factorize A = L*U using GE (cost = 2/3 n^3 flops)
  - Solve L*y = b for y, using substitution (cost = n^2 flops)
  - Solve U*x = y for x, using substitution (cost = n^2 flops)
- Thus A*x = (L*U)*x = L*(U*x) = L*y = b as desired

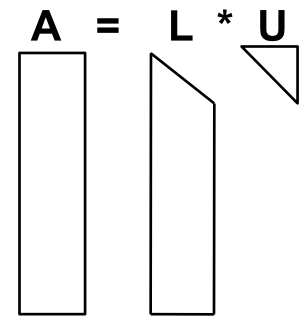
Problems with basic GE algorithm

```
for i = 1 to n-1
    A(i+1:n,i) = A(i+1:n,i) / A(i,i)          … BLAS 1 (scale a vector)
    A(i+1:n,i+1:n) = A(i+1:n , i+1:n )  … BLAS 2 (rank-1 update)
            - A(i+1:n , i) * A(i , i+1:n)
```

- What if some A(i,i) is zero? Or very small?
  - Result may not exist, or be "unstable", so need to pivot
- Current computation all BLAS 1 or BLAS 2, with low computational intensities ($q \le 2$), need to use BLAS 3 operations like matmul instead



Recursive, cache-oblivious GE formulation (1/3)



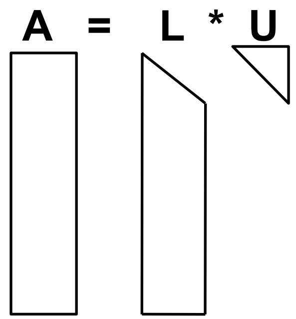
- Toledo (1997)
  - Describe without pivoting for simplicity
  - "Do left half of matrix, then right half"

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I \end{pmatrix} * \begin{pmatrix} U_{11} & U_{12} \\ 0 & S \end{pmatrix} = \begin{pmatrix} L_{11}*U_{11} & L_{11}*U_{12} \\ L_{21}*U_{11} & S+L_{21}*U_{12} \end{pmatrix}$$

- **Four step recursive algorithm**
  1. Factor $\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} L_{11}*U_{11} \\ L_{21}*U_{11} \end{pmatrix} = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} * U_{11}$ same problem with half the columns: solve recursively
  2. Solve $A_{12} = L_{11}*U_{12}$ for U_{12}; call triangular solve (BLAS3)
  3. Solve $A_{22} = S+L_{21}*U_{12}$ for $S = A_{22} - L_{21}*U_{12}$; matmul (BLAS3)
  4. Factor $S = L_{22}*U_{22}$; same problem with half the columns, fewer rows: solve recursively

Recursive, cache-oblivious GE formulation (2/3)

- Toledo (1997)
  - Describe without pivoting for simplicity
  - "Do left half of matrix, then right half"



function [L,U] = RLU (A) … assume A is m by n
if (n=1) L = A/A(1,1), U = A(1,1)
else
[L1,U1] = RLU(A(1:m , 1:n/2)) … do left half of A
… let L11 denote top n/2 rows of L1
A(1:n/2 , n/2+1 : n) = L11^{-1} * A(1:n/2 , n/2+1 : n)
… update A12 (top n/2 rows of right half of A)
A(n/2+1: m, n/2+1:n) = A(n/2+1: m, n/2+1:n)
- A(n/2+1: m, 1:n/2) * A(1:n/2 , n/2+1 : n)
… update rest of right half of A, get S
[L2,U2] = RLU(A(n/2+1:m , n/2+1:n)) … do right half of A
return [L1,[0;L2]] and [U1, [A(.,.) ; U2]]

Alternative cache-oblivious GE formulation (3/3)

```
function [L,U] = RLU (A)   … assume A is m by n
    if (n=1)   L = A/A(1,1),  U = A(1,1)
    else
        [L1,U1] = RLU( A(1:m , 1:n/2)) … do left half of A
            … let L11 denote top n/2 rows of L1
        A( 1:n/2 , n/2+1 : n ) = L11^-1 * A( 1:n/2 , n/2+1 : n )
            … update A12 (top n/2 rows of right half of A)
        A( n/2+1: m, n/2+1:n ) = A( n/2+1: m, n/2+1:n )
            - A( n/2+1: m, 1:n/2 ) * A( 1:n/2 , n/2+1 : n )
            … update rest of right half of A, get S
        [L2,U2] = RLU( A(n/2+1:m , n/2+1:n) ) … do right half of A
        return [ L1,[0;L2] ] and [U1, [ A(.,.) ; U2 ] ]
```

- Performs same flops as original algorithm, just in a different order
- $W(m,n)$ = # words moved to factor m-by-n matrix

$= W(m,n/2) + O(\max(m \cdot n, m \cdot n^2/M^{1/2})) + W(m-n/2,n/2)$

$\leq 2 \cdot W(m,n/2) + O(\max(m \cdot n, m \cdot n^2/M^{1/2}))$

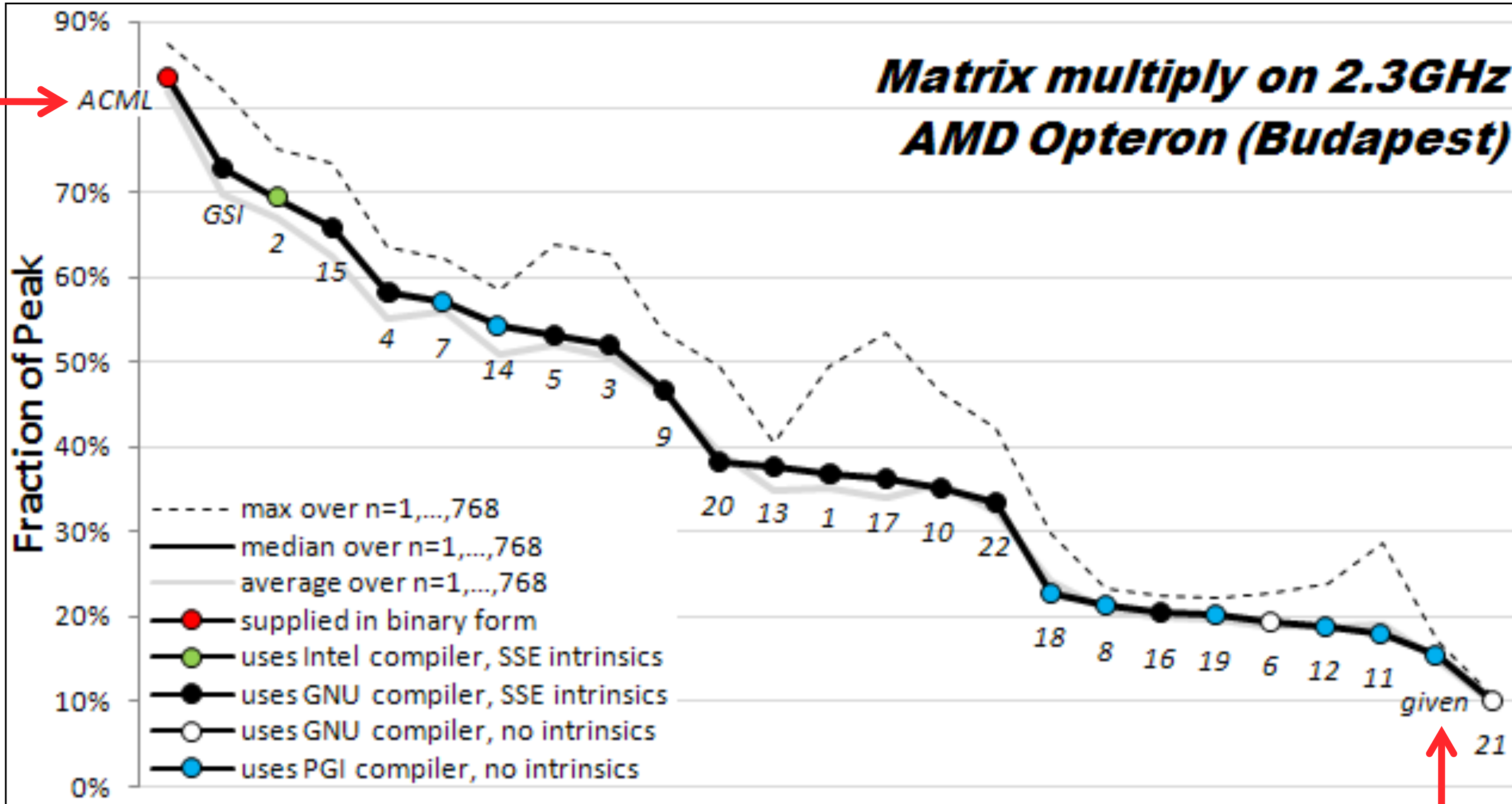
$= O(m \cdot n^2/M^{1/2} + m \cdot n \cdot \log M)$

$= O(m \cdot n^2/M^{1/2})$ if $M^{1/2} \cdot \log M = O(n)$, i.e. usually attains lower bound

Other Topics

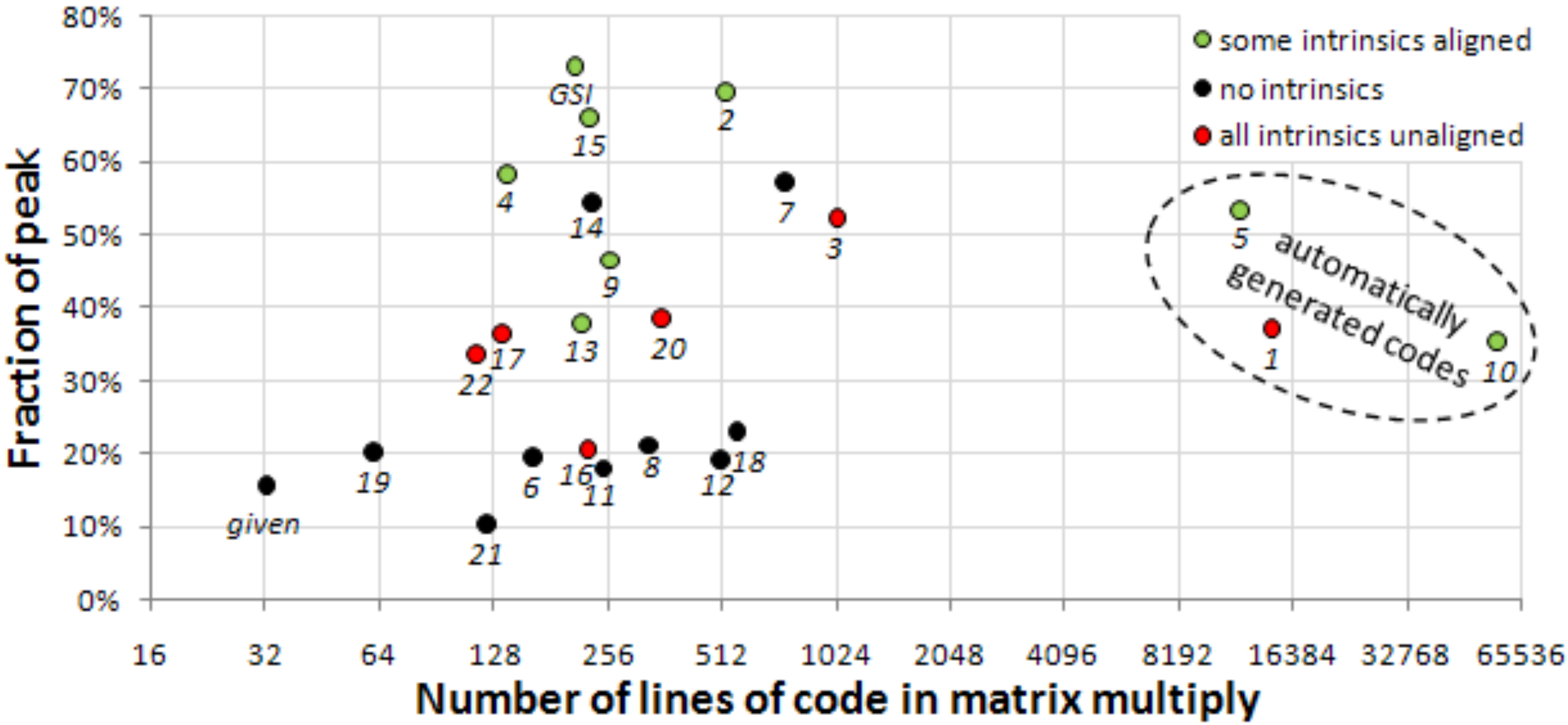
- Cache oblivious algorithms not a panacea
  - Recursing down to problems of size 1 add too much overhead
  - Need to switch to blocked algorithm for small enough subproblems
- Algorithms for other linear algebra problems (eg least squares, eigenvalue problems) more complicated
  - Minimizing communication requires different mathematical algorithms, not just different order of execution
- Dense linear algebra possible in $O(n^w)$ flops, with $w < 3$
  - Eg: Strassen's algorithm has $w = \log_2 7 \sim 2.81$
  - Less communication too
- Only a few algorithms known in sparse case that minimize communication
- See Ma221, CS267 for details

How hard is hand-tuning matmul, anyway?



- Results of 22 student teams trying to tune matrix-multiply, in CS267 Spr09
- Students given "blocked" code to start with
  - Still hard to get close to vendor tuned performance (ACML)
- For more discussion, see www.cs.berkeley.edu/~volkov/cs267.sp09/hw1/results/

How hard is hand-tuning matmul, anyway?



Industrial-strength dense linear algebra

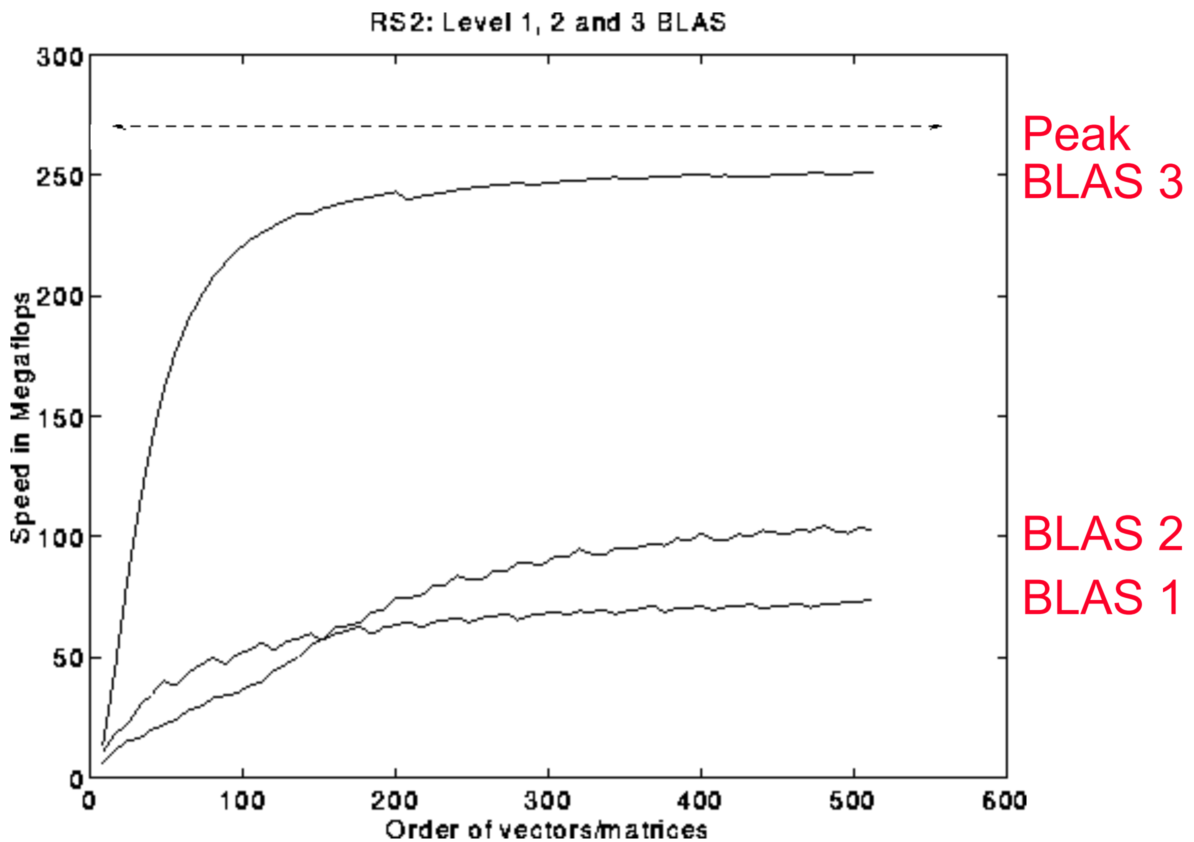
- **Uses the correct abstractions to formulate the algorithms (block matrix-matrix multiply)**
- **Software building blocks so that those algorirhms can be implemented efficiently with high performance (BLAS)**
- **Machines are really complicated, so some choices of algorithm parameters must be discovered empirically (autotuning).**

Basic Linear Algebra Subroutines (BLAS)

- Industry standard interface (evolving)
  - www.netlib.org/blas, www.netlib.org/blas/blast--forum
- Vendors, others supply optimized implementations
- History
  - BLAS1 (1970s):
    - vector operations: dot product, saxpy ($y=\alpha*x+y$), etc
    - m=2*n, f=2*n, q ~1 or less
  - BLAS2 (mid 1980s)
    - matrix-vector operations: matrix vector multiply, etc
    - m=n^2, f=2*n^2, q~2, less overhead
    - somewhat faster than BLAS1
  - BLAS3 (late 1980s)
    - matrix-matrix operations: matrix matrix multiply, etc
    - m <= 3n^2, f=O(n^3), so q=f/m can possibly be as large as n, so BLAS3 is potentially much faster than BLAS2
- Good algorithms use BLAS3 when possible (LAPACK & ScaLAPACK)

BLAS speeds on an IBM RS6000/590

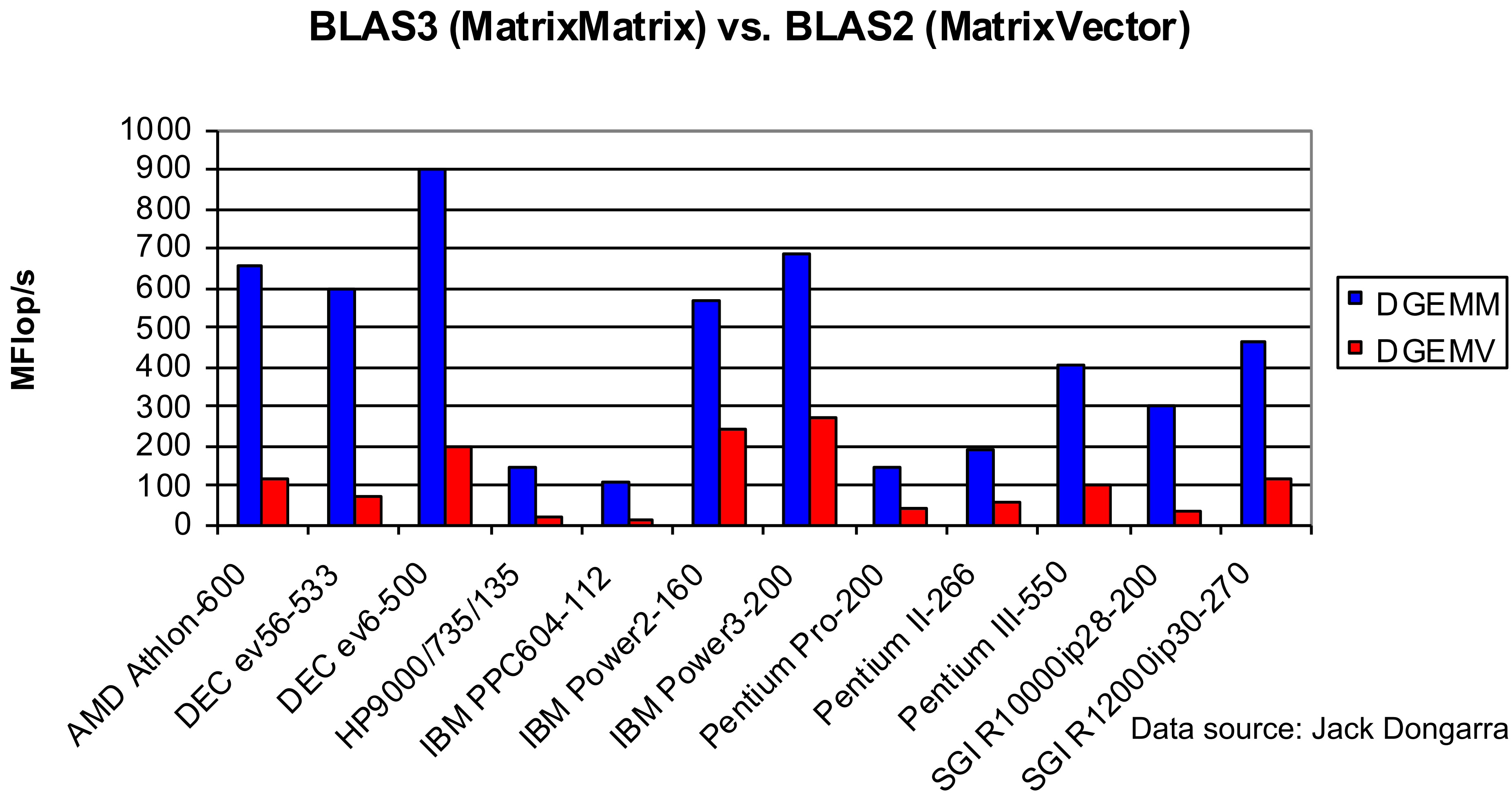
Peak speed = 266 Mflops



BLAS 3 (n-by-n matrix matrix multiply) vs
BLAS 2 (n-by-n matrix vector multiply) vs
BLAS 1 (saxpy of n vectors)

Dense Linear Algebra: BLAS2 vs. BLAS3

- BLAS2 and BLAS3 have very different computational intensity, and therefore different performance



Data source: Jack Dongarra

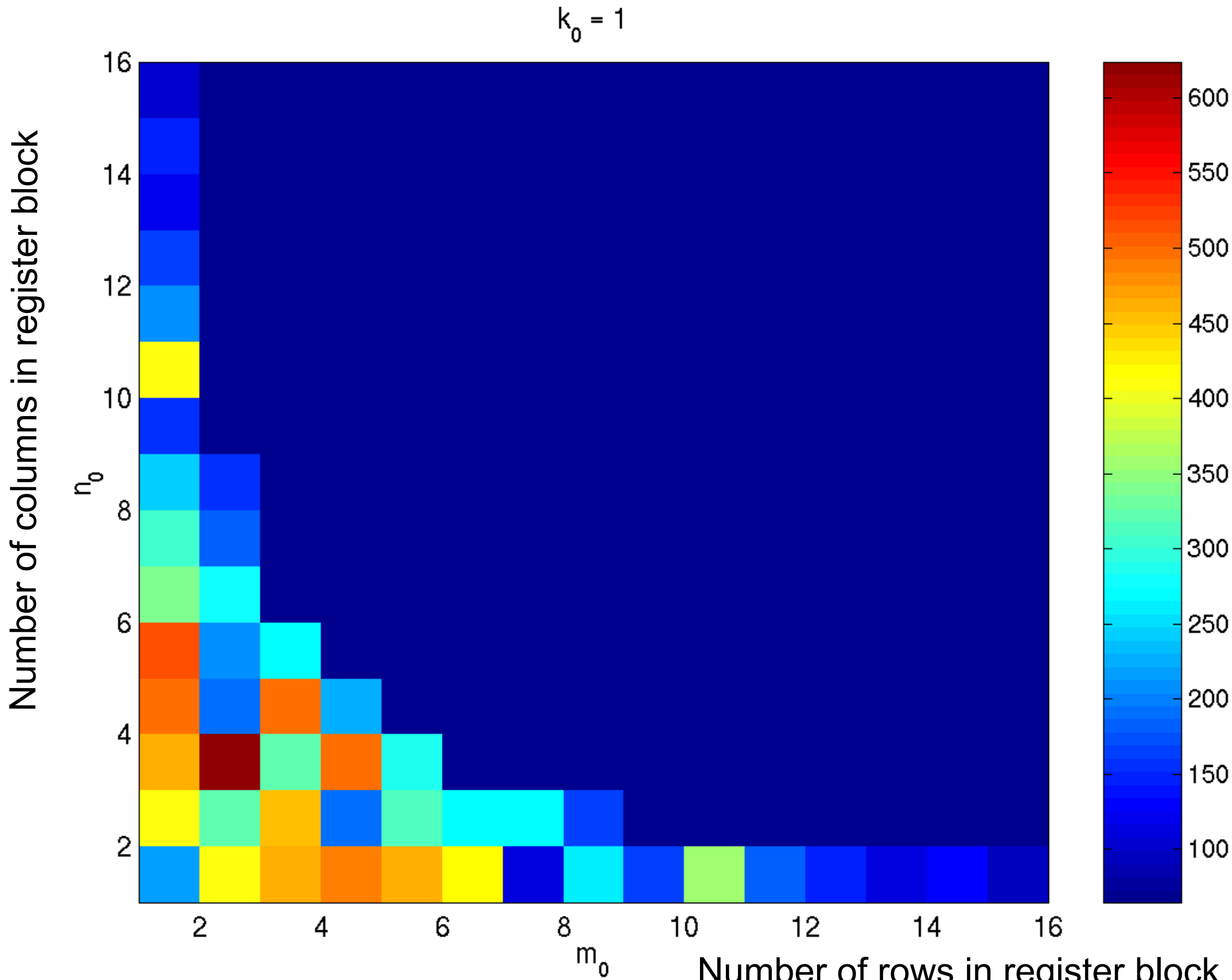
Tuning Code in Practice

- Tuning code can be tedious
  - Lots of code variations to try besides blocking
  - Machine hardware performance hard to predict
  - Compiler behavior hard to predict
- Response: "Autotuning"
  - Let computer generate large set of possible code variations, and search them for the fastest ones
  - Field started with CS267 homework assignment in mid 1990s
    - PHiPAC, leading to ATLAS, incorporated in Matlab
    - We still use the same assignment
  - We (and others) are extending autotuning to other motifs
- Still need to understand how to do it by hand
  - Not every code will have an autotuner
  - Need to know if you want to build autotuners

Search Over Block Sizes

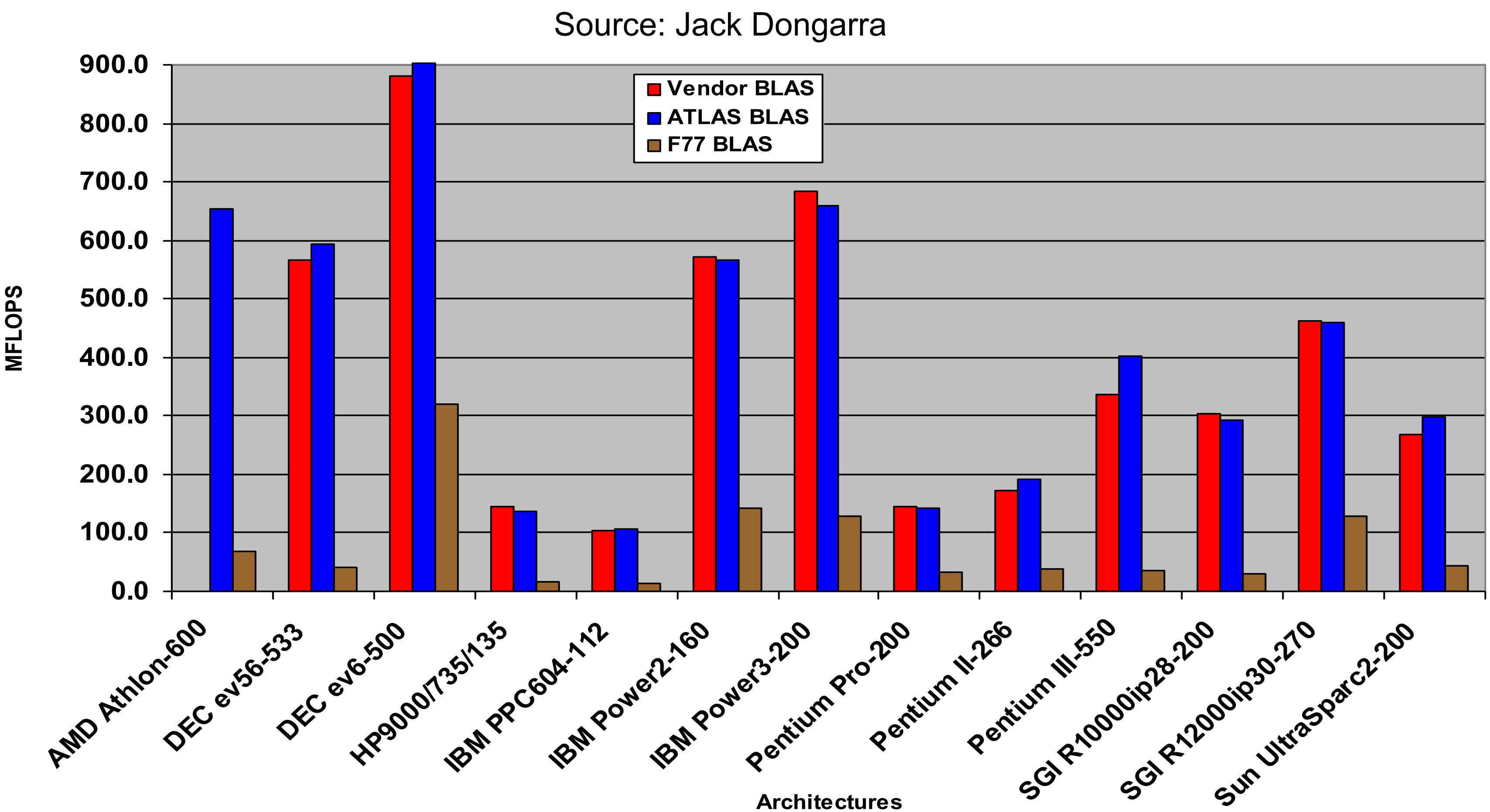
- Performance models are useful for high level algorithms
  - Helps in developing a blocked algorithm
  - Models have not proven very useful for block size selection
    - too complicated to be useful
    - too simple to be accurate
      - Multiple multidimensional arrays, virtual memory, etc.
  - Speed depends on matrix dimensions, details of code, compiler, processor

What the Search Space Looks Like



A 2-D slice of a 3-D register-tile search space. The dark blue region was pruned.
(Platform: Sun Ultra-IIi, 333 MHz, 667 Mflop/s peak, Sun cc v5.0 compiler)

ATLAS (DGEMM n = 500)



- ATLAS is faster than all other portable BLAS implementations and it is comparable with machine-specific libraries provided by the vendor.