61A Lecture 6

Announcements

Recursive Functions

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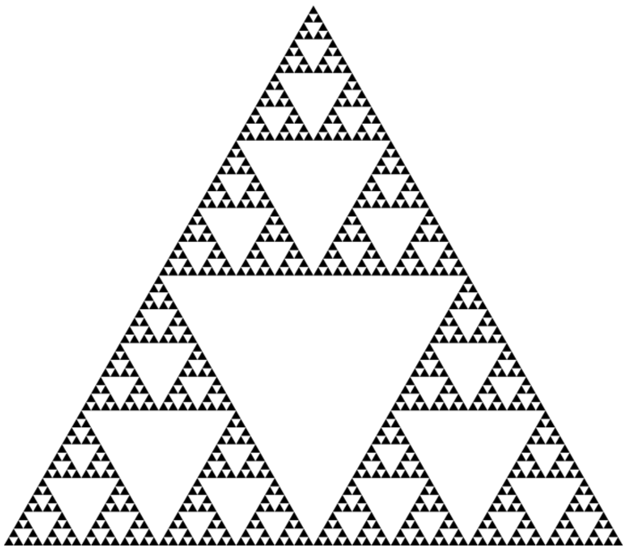

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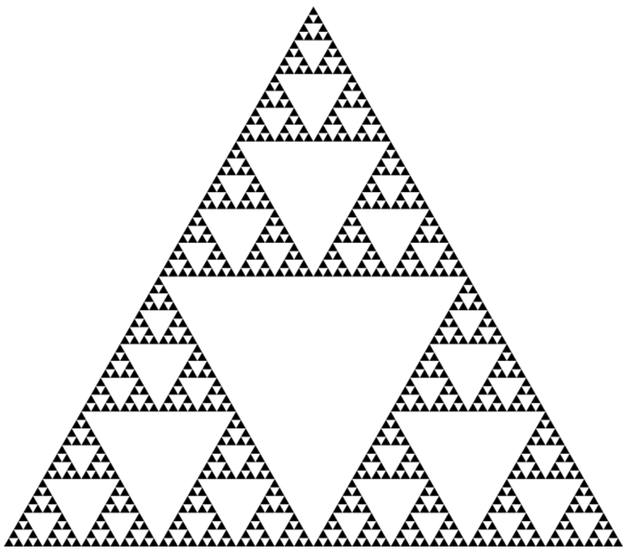

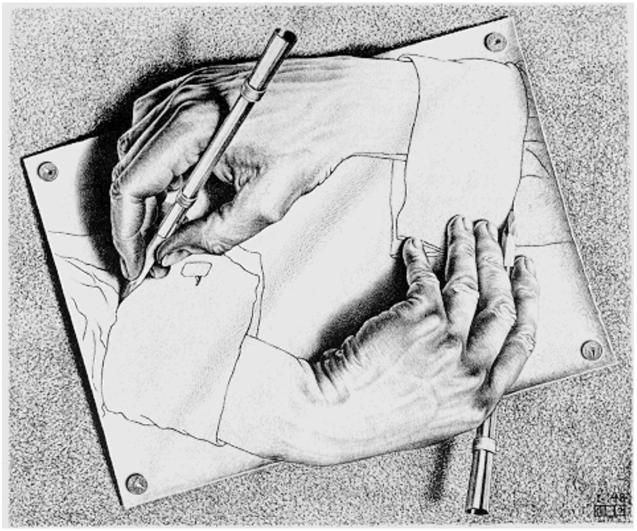

Drawing Hands, by M. C. Escher (lithograph, 1948)

Digit Sums

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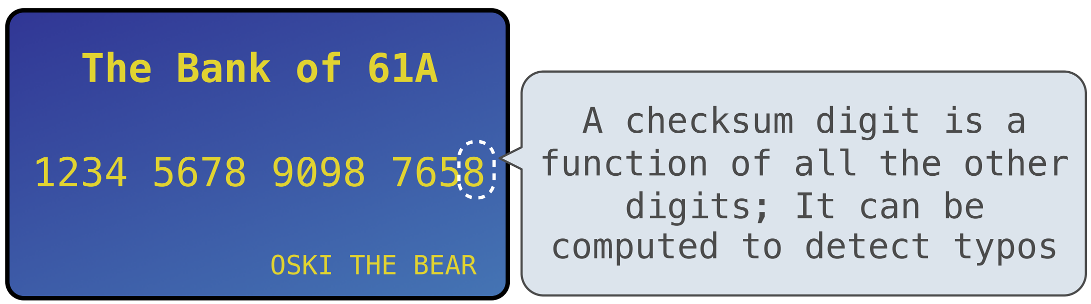

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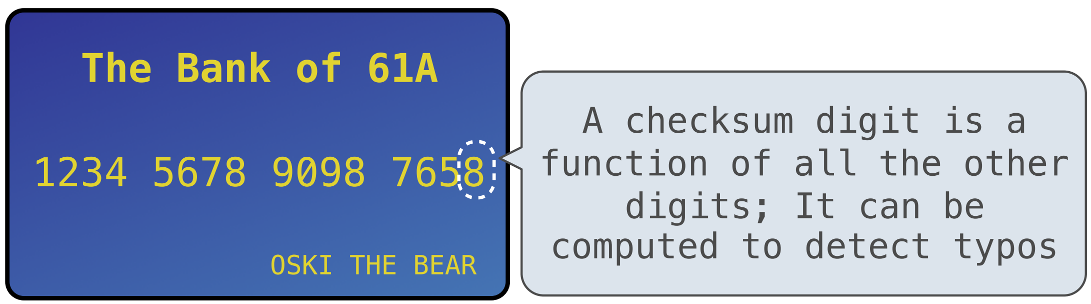

- Credit cards actually use the Luhn algorithm, which we'll implement after digit_sum

Sum Digits Without a While Statement

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Recursion in Environment Diagrams

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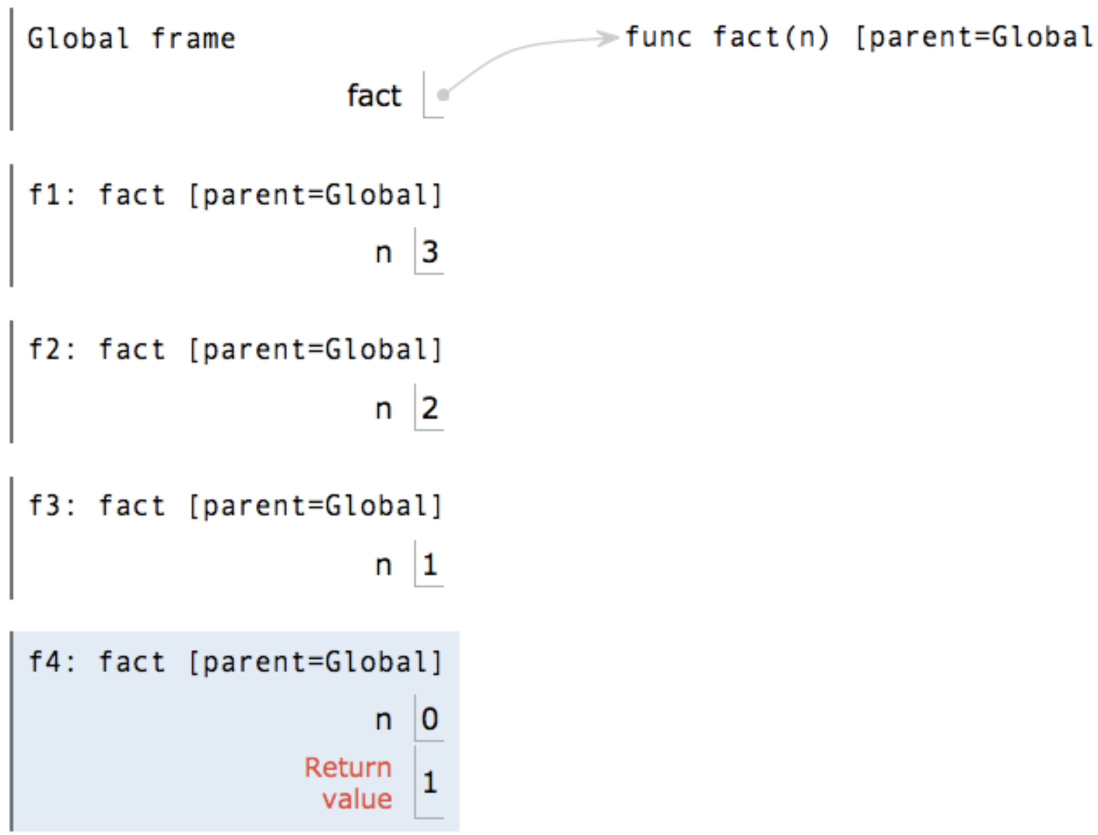

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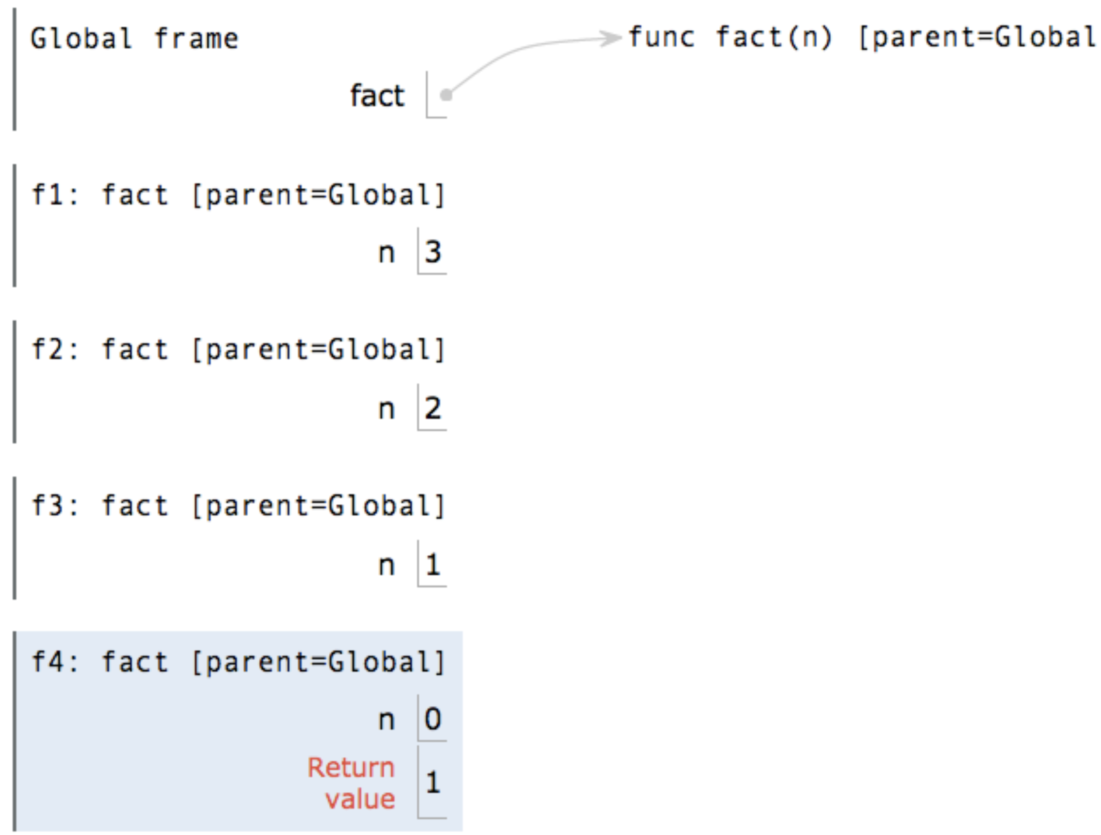

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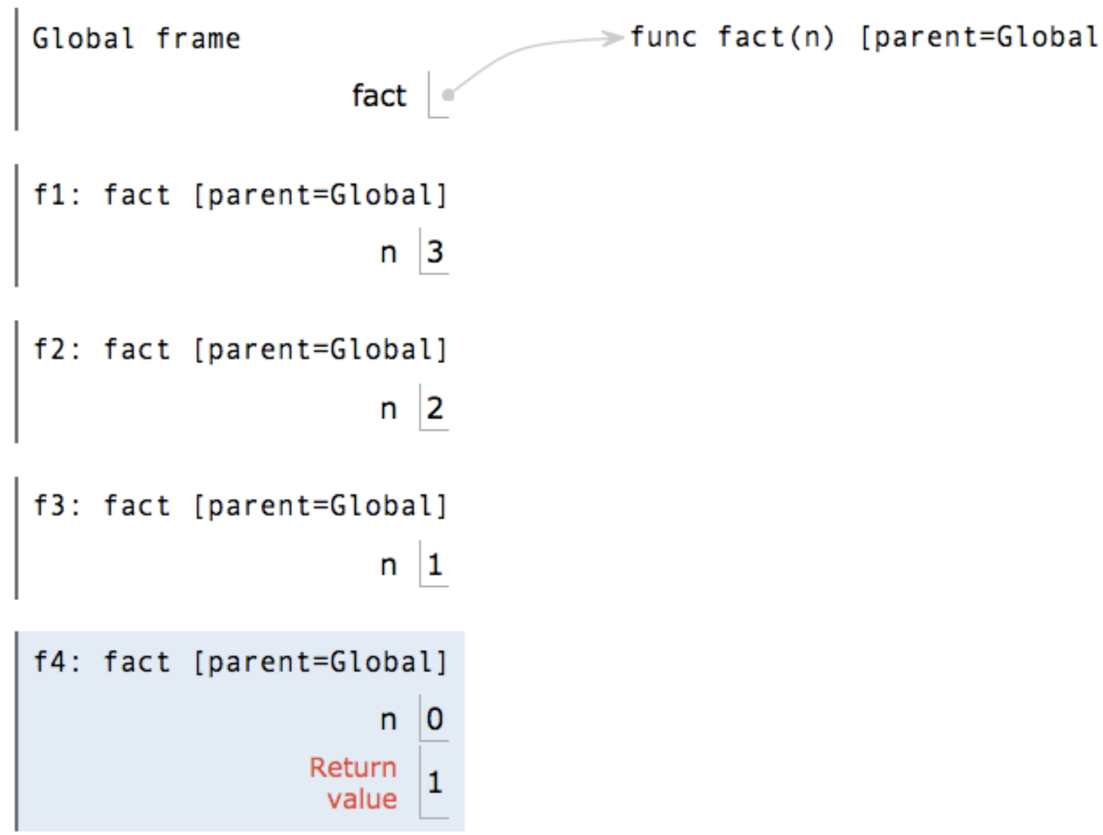

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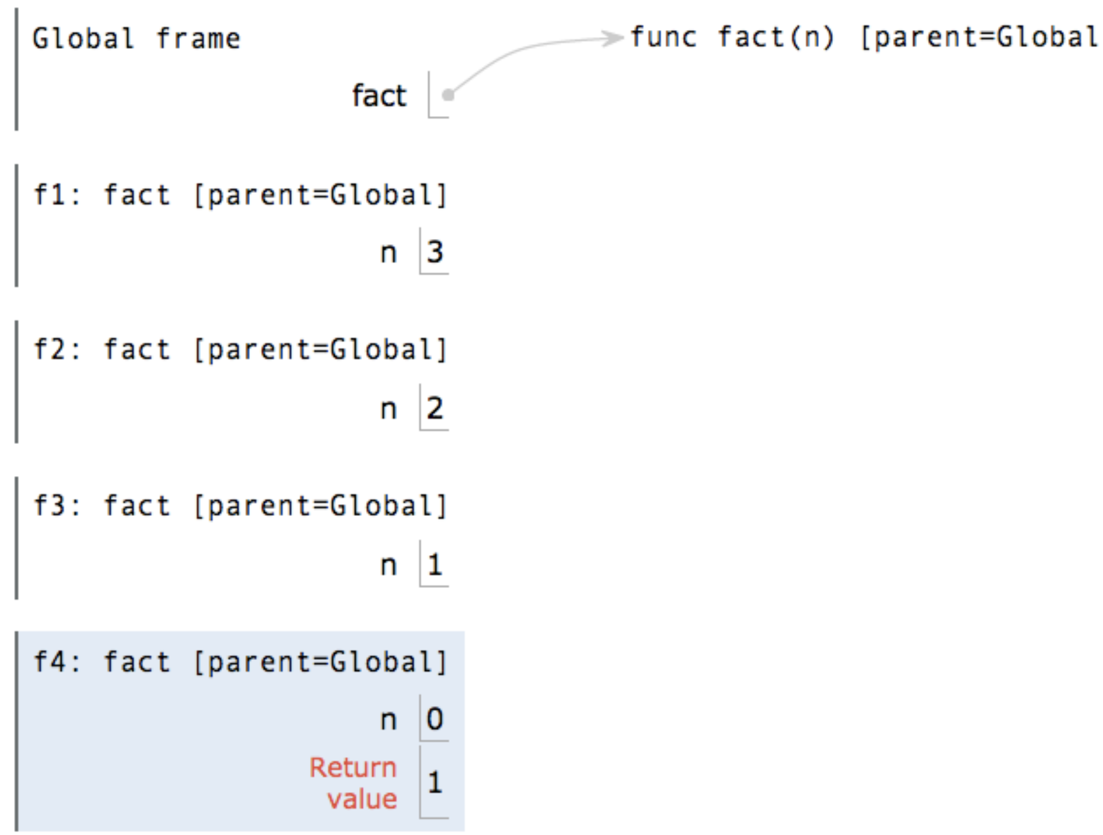

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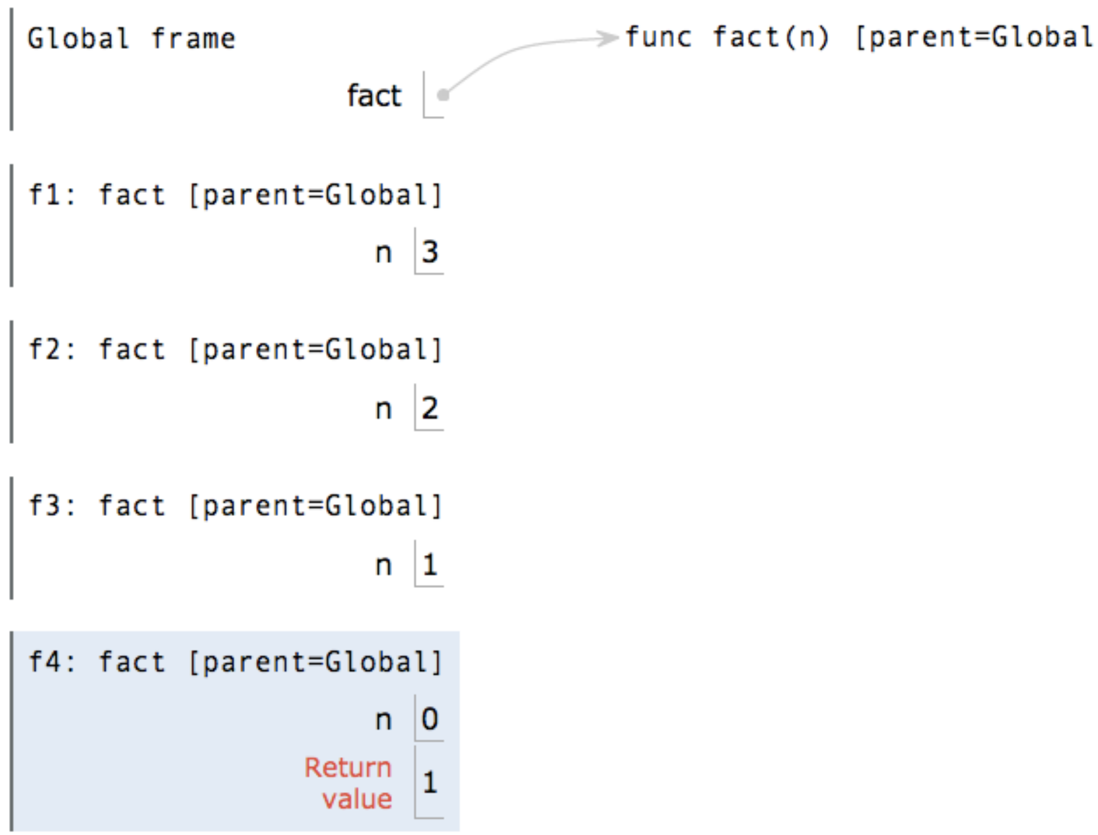

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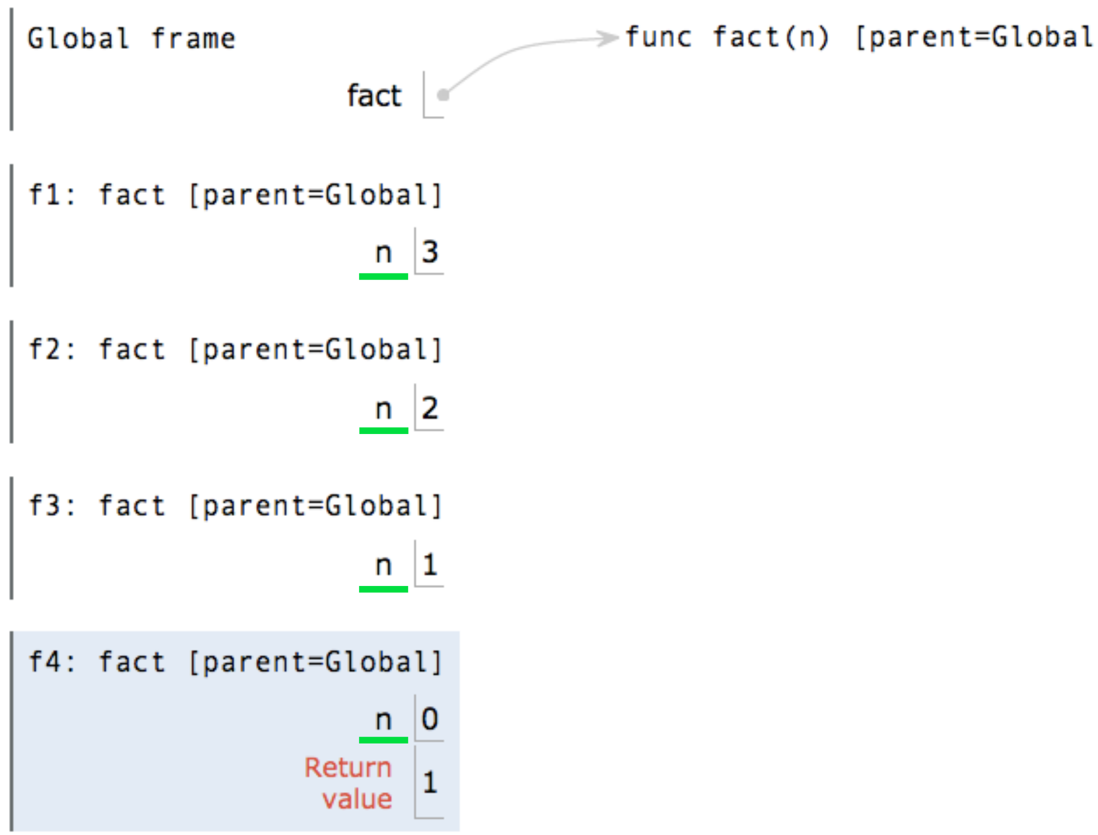

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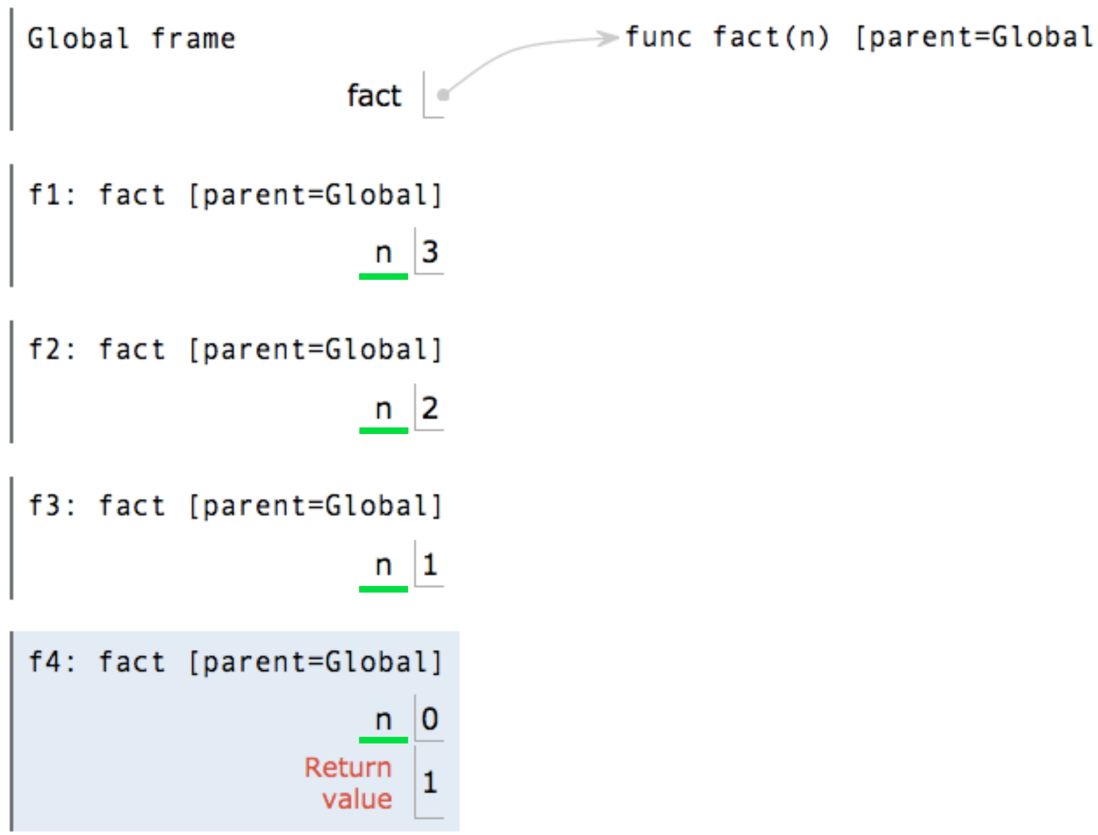

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- Each call to **fact** solves a simpler problem than the last: smaller **n**

Interactive Diagram

Iteration vs Recursion

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n, fact

Verifying Recursive Functions

The Recursive Leap of Faith

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Photo by Kevin Lee, Preikestolen, Norway

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Is fact implemented correctly?

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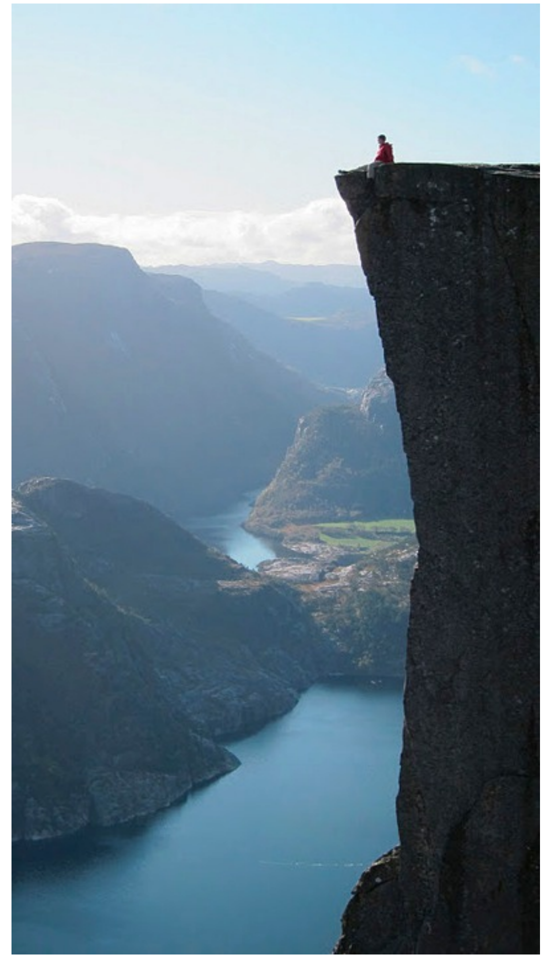

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2. Treat **fact** as a functional abstraction!
3. Assume that **fact(n-1)** is correct
4. Verify that **fact(n)** is correct

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Mutual Recursion

The Luhn Algorithm

The Luhn Algorithm

Used to verify credit card numbers

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From Wikipedia: http://en.wikipedia.org/wiki/Luhn_algorithm

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The Luhn sum of a valid credit card number is a multiple of 10 (Demo)

Recursion and Iteration

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A partial sum

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Converting Recursion to Iteration

Can be tricky: Iteration is a special case of recursion.

Idea: Figure out what state must be maintained by the iterative function.

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

A partial sum

What's left to sum

(Demo)

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```
def sum_digits_iter(n):
    digit_sum = 0
    while n > 0:
        n, last = split(n)
        digit_sum = digit_sum + last
    return digit_sum
```

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```
def sum_digits_rec(n, digit_sum):
    if n == 0:
        return digit_sum
    else:
        n, last = split(n)
        return sum_digits_rec(n, digit_sum + last)
```

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def sum_digits_iter(n):
    digit_sum = 0
    while n > 0:
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        digit_sum = digit_sum + last
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Updates via assignment become...

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def sum_digits_rec(n, digit_sum):
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        return digit_sum
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def sum_digits_rec(n, digit_sum):
    if n == 0:
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```

...arguments to a recursive call