Recursion

Announcements

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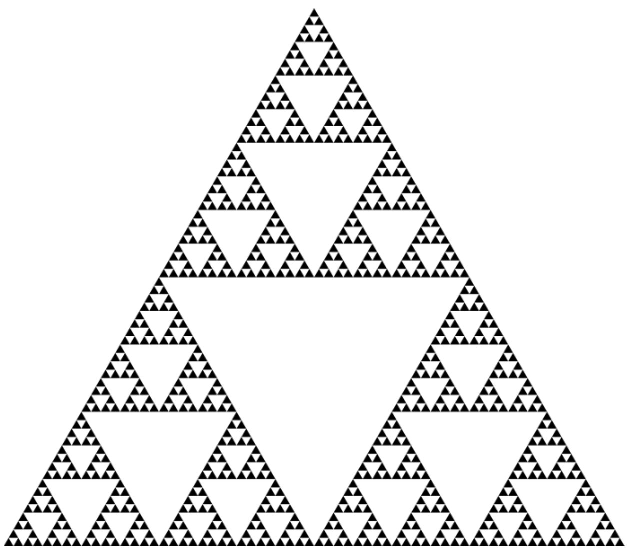

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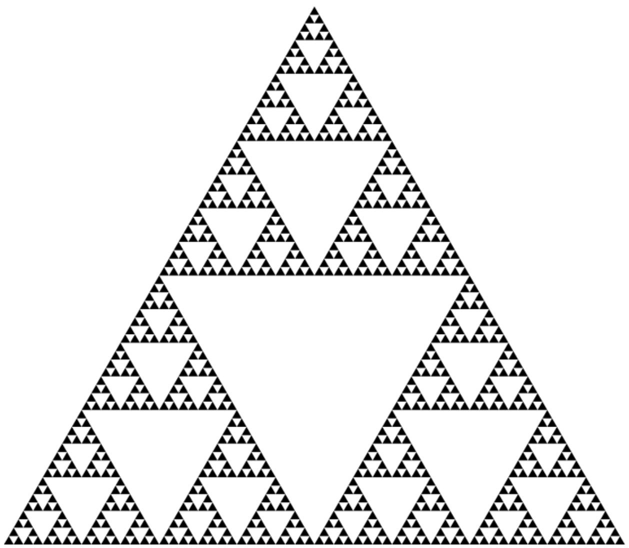

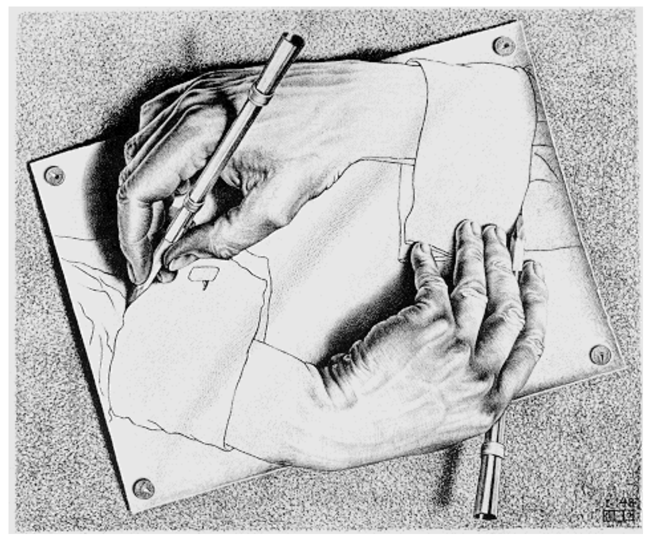

Drawing Hands, by M. C. Escher (lithograph, 1948)

Digit Sums

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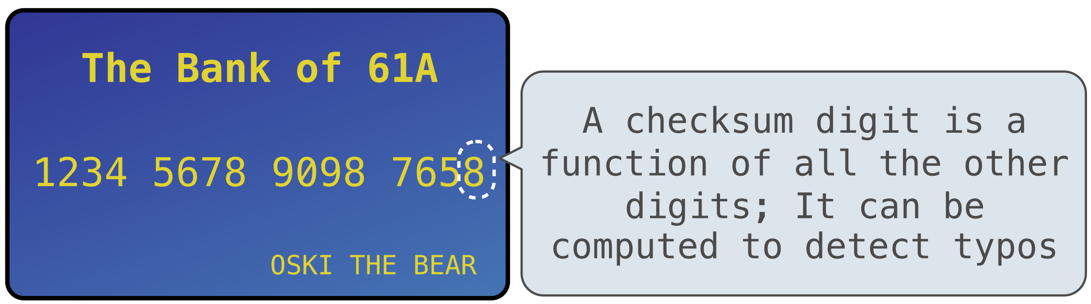

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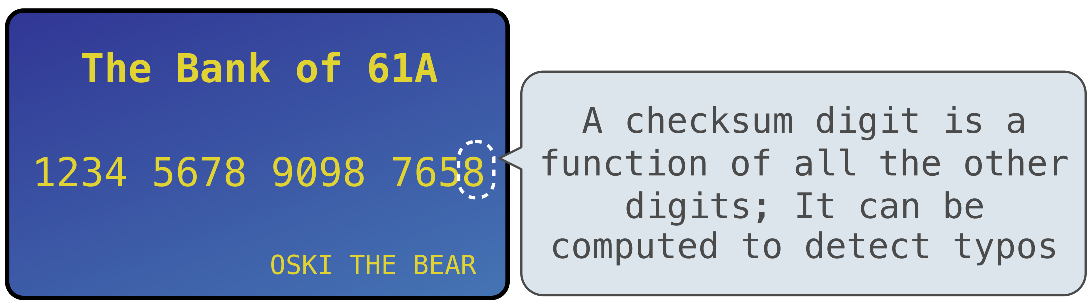


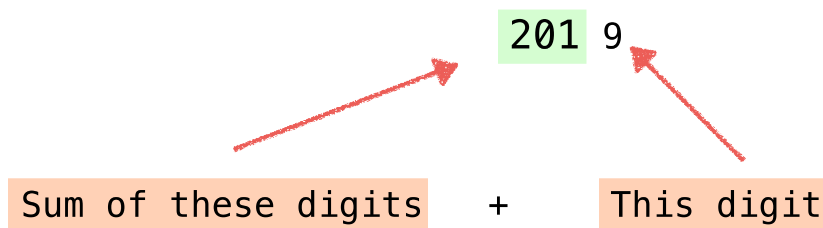
- Credit cards actually use the Luhn algorithm, which we'll implement after sum_digits

The Problem Within the Problem

The sum of the digits of 6 is 6.

Likewise for any one-digit (non-negative) number (i.e., < 10).

The sum of the digits of 2019 is



That is, we can break the problem of summing the digits of 2019 into a smaller instance of the same problem, plus some extra stuff.

We call this recursion

Sum Digits Without a While Statement

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(Demo)

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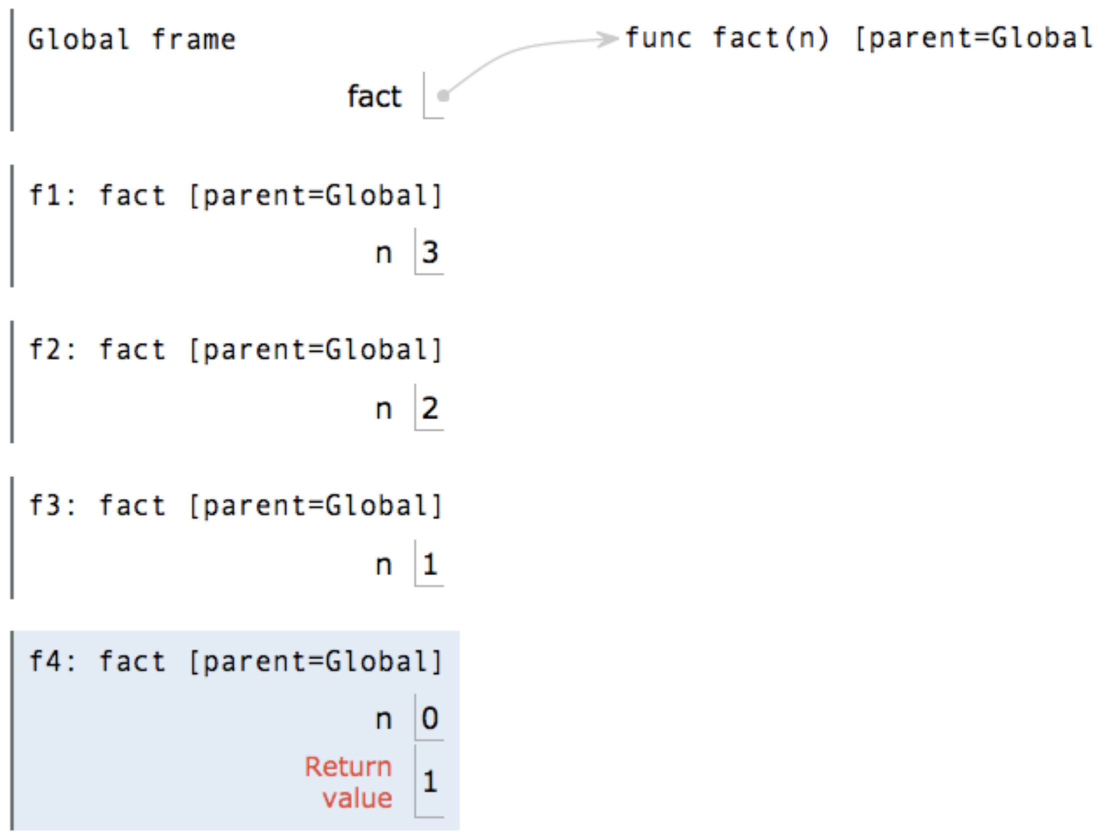

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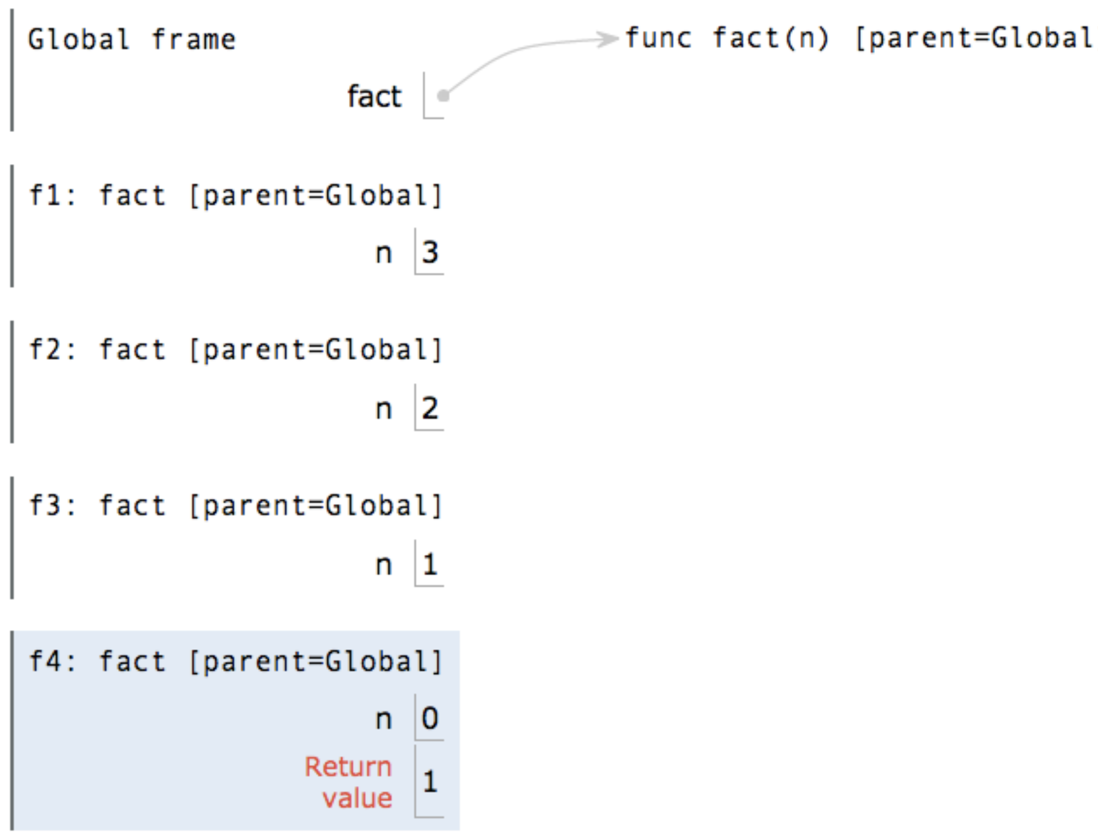

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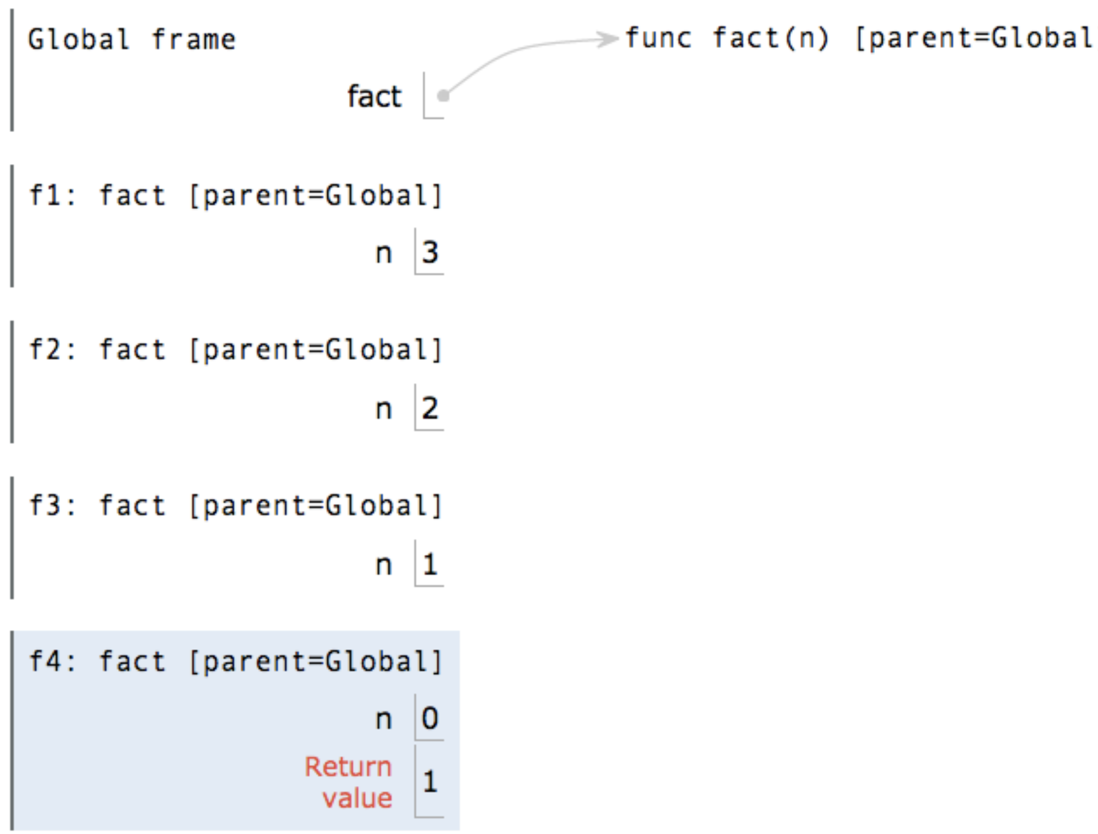

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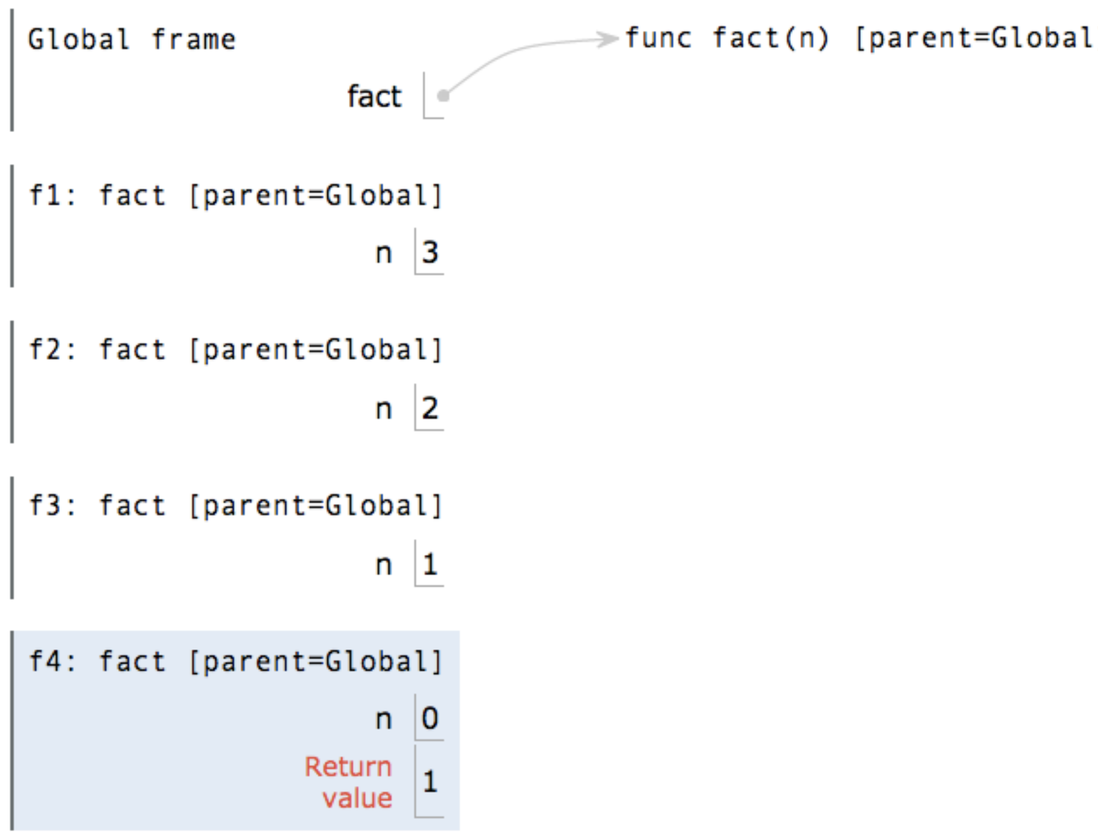

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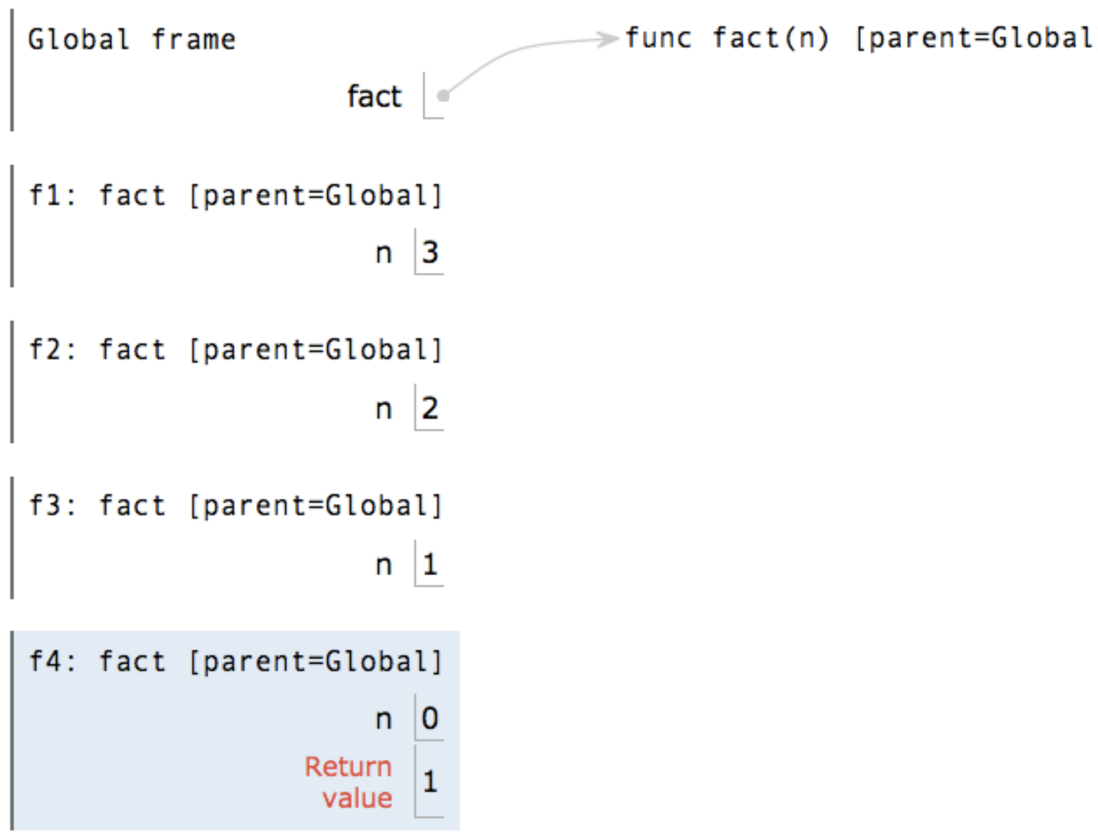

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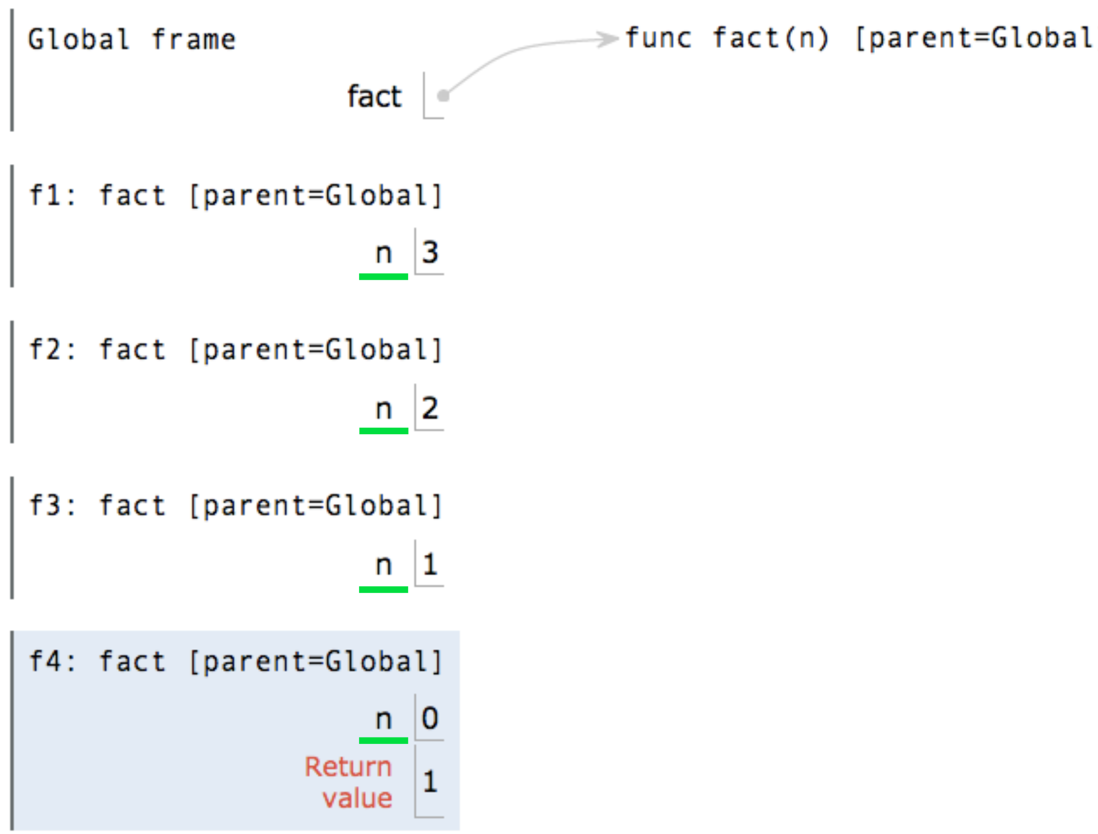

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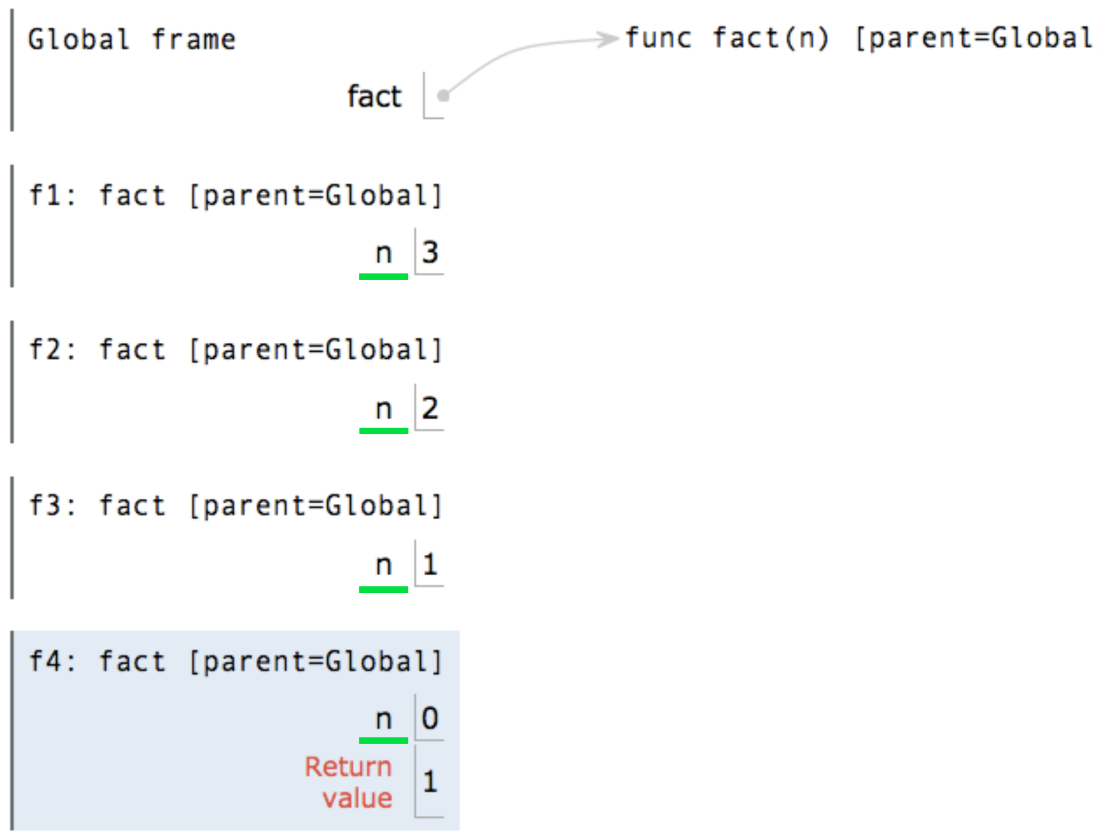

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- Each call to **fact** solves a simpler problem than the last: smaller **n**

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n, fact

Verifying Recursive Functions

The Recursive Leap of Faith

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Photo by Kevin Lee, Preikestolen, Norway

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        return n * fact(n-1)
```

Is fact implemented correctly?

1. Verify the base case

2. Treat **fact** as a functional abstraction!

3. Assume that **fact(n-1)** is correct

Photo by Kevin Lee, Preikestolen, Norway

The Recursive Leap of Faith

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def fact(n):
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        return n * fact(n-1)
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4. Verify that **fact**(n) is correct

Photo by Kevin Lee, Preikestolen, Norway

Mutual Recursion

The Luhn Algorithm

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Used to verify credit card numbers

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From Wikipedia: http://en.wikipedia.org/wiki/Luhn_algorithm

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- **First:** From the rightmost digit, which is the check digit, moving left, double the value of every second digit; if product of this doubling operation is greater than 9 (e.g., 7 * 2 = 14), then sum the digits of the products (e.g., 10: 1 + 0 = 1, 14: 1 + 4 = 5)

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Recursion and Iteration

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def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
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...arguments to a recursive call