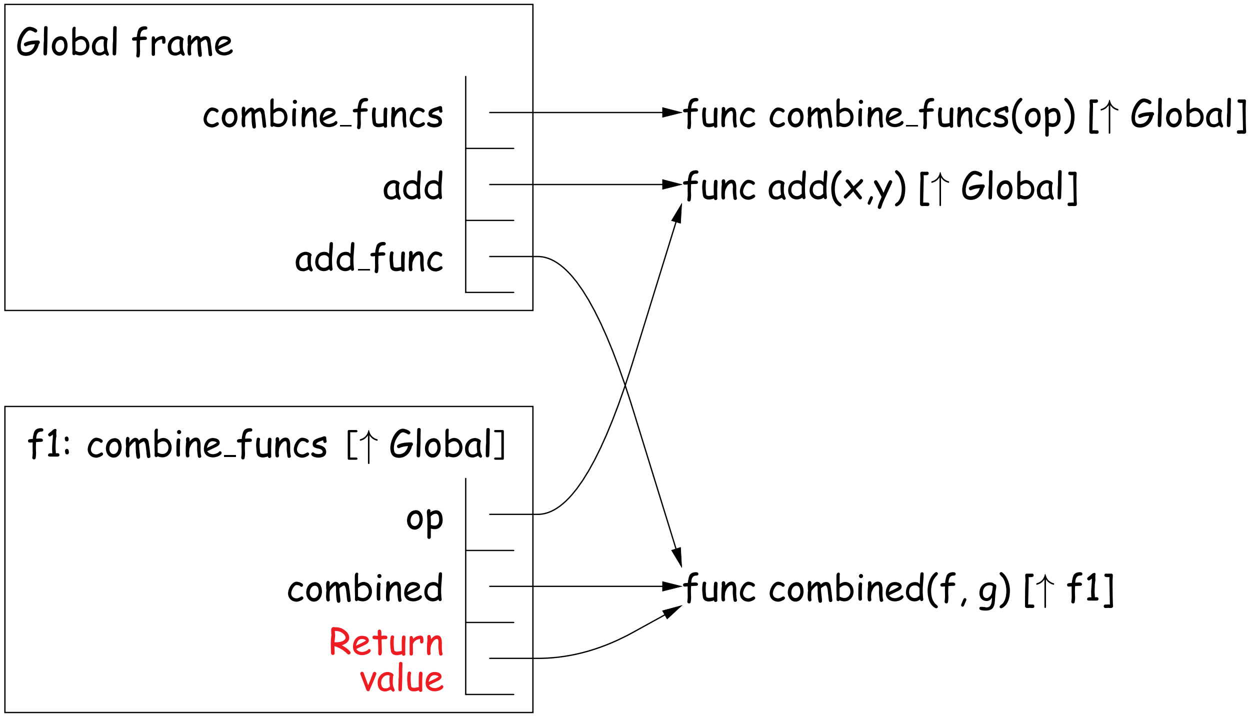
Lecture #5: Higher-Order Functions

Example From Last Time (I)

```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
```



Global frame

combine_funcs → func combine_funcs(op) [↑ Global]

add → func add(x,y) [↑ Global]

add_func → func combined(f, g) [↑ f1]

f1: combine_funcs [↑ Global]

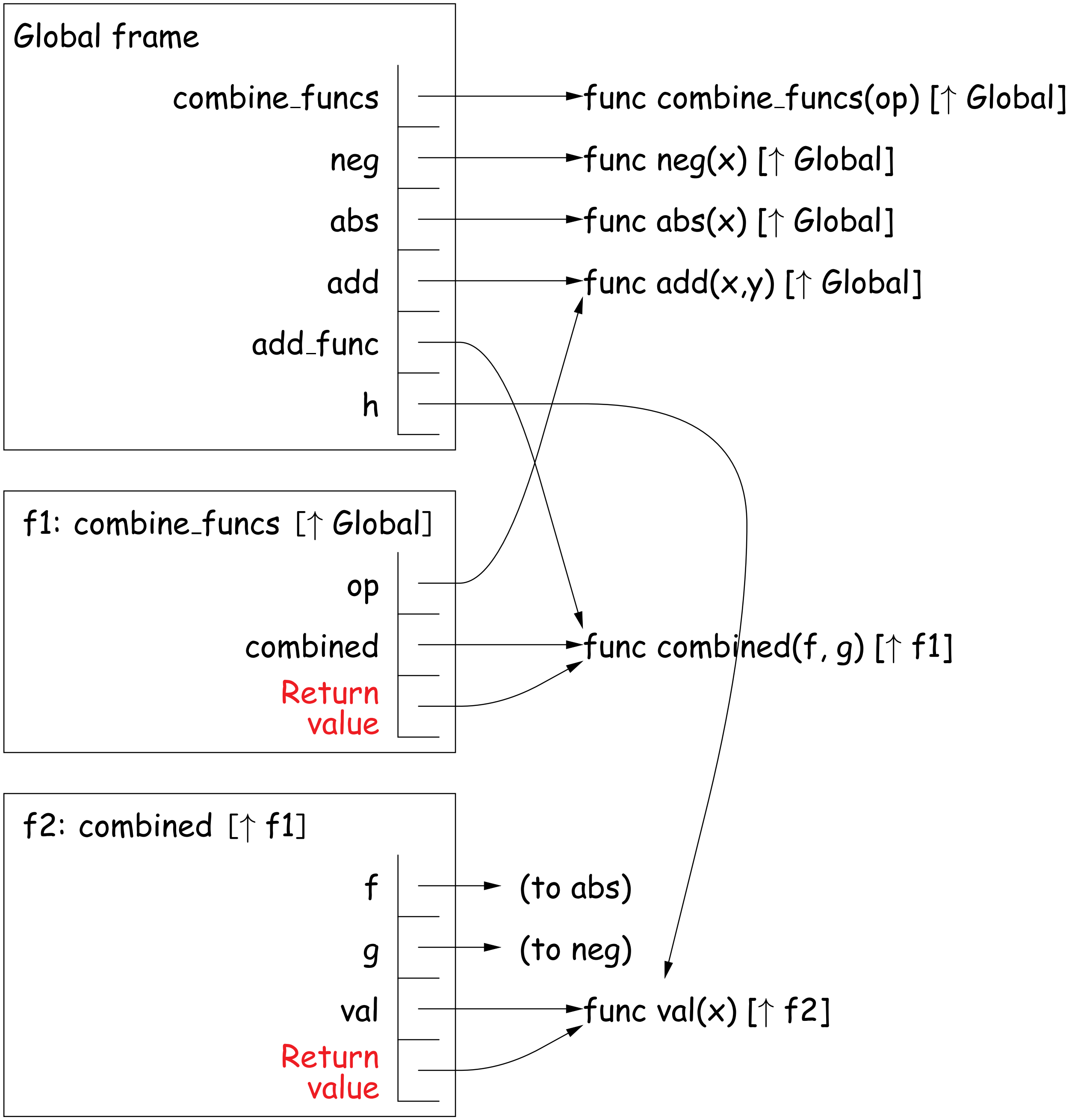
op → func add(x,y) [↑ Global]

combined → func combined(f, g) [↑ f1]

Return value → func combined(f, g) [↑ f1]

Example From Last Time (II)

```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
h = add_func(abs, neg)
```



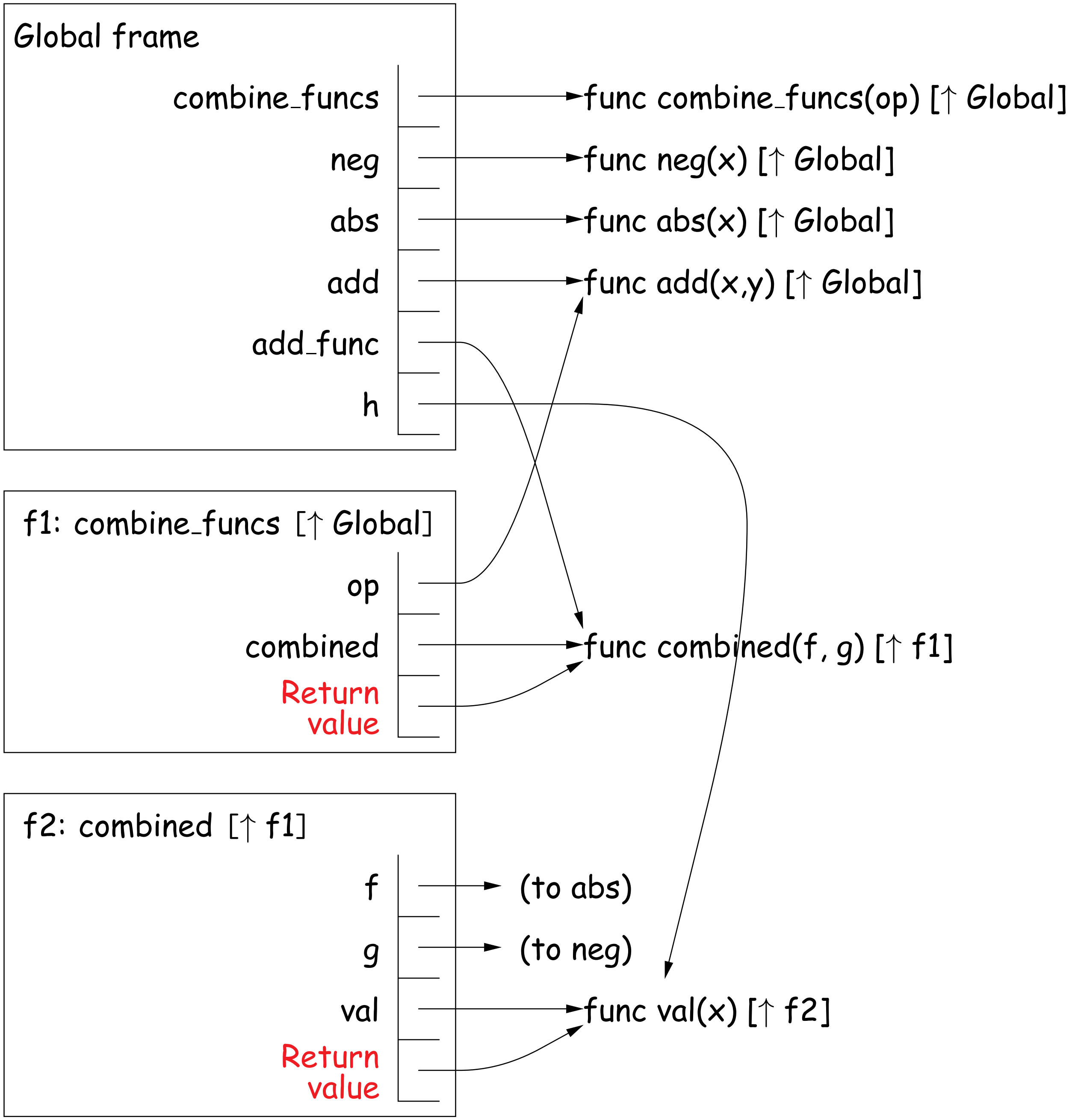
Global frame
combine_funcs → func combine_funcs(op) [↑ Global]
neg → func neg(x) [↑ Global]
abs → func abs(x) [↑ Global]
add → func add(x,y) [↑ Global]
add_func
h

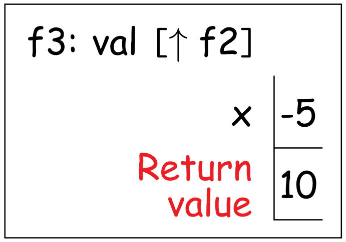
f1: combine_funcs [↑ Global]
op
combined → func combined(f, g) [↑ f1]
Return value

f2: combined [↑ f1]
f → (to abs)
g → (to neg)
val → func val(x) [↑ f2]
Return value

Example From Last Time (III)

```
def combine_funcs(op):
  def combined(f, g):
    def val(x):
      return op(f(x), g(x))
    return val
  return combined
add_func = combine_funcs(add)
h = add_func(abs, neg)
h(-5)
```





. . . and local frames for

- add (value of op),
- abs (value of f), and
- neg (value of g)

Do You Understand the Machinery? (IV)

What is printed: (1, infinite loop, or error) and why?

```
def g(x):
    print(x)

def f(f):
    f(1)

f(g)
```

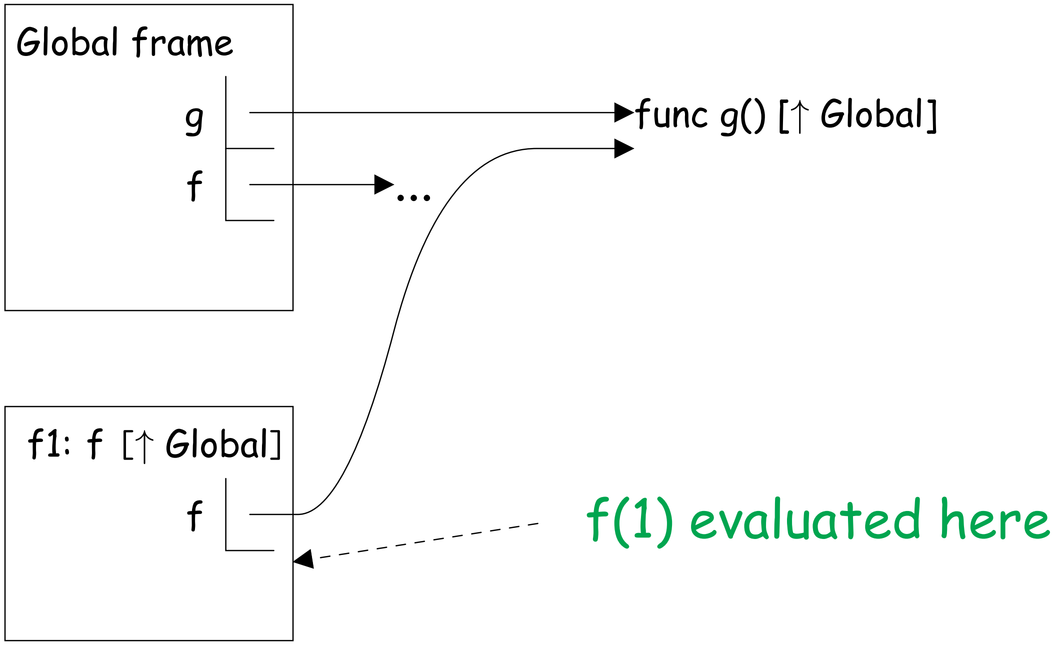
Answer (IV)

This prints 1. When we reach `f(1)` inside `f`, the call expression, and therefore the name `f`, evaluated in the environment E, where the value of `f` is the global function bound to `g`:

```
def g(x):
    print(x)

def f(f):
    f(1)

f(g)
```



Do You Understand the Machinery? (V)

What is printed: (0, 1, or error) and why?

```
def f():
    return 0

def g():
    return f()

def h(k):
    def f():
        return 1
    p = k
    return p()

print(h(g))
```

Answer (V)

This prints 0. Function values are attached to current environments when they are first created (by lambda or def). Assignments (such as to p) don't themselves create new values, but only copy old ones, so that when p is evaluated, it is equal to k, which is equal to g, which is attached to the global environment.

Observation: Environments Reflect Nesting

- From what we've seen so far:

  Linking of environment frames $\Longleftrightarrow$ Nesting of definitions.

- For example, given

  ```
  def f(x):
      def g(x):
          def h(x):
              print(x)
          ...
      ...
  ```

  The structure of the program tells you that the environment in which *print(x)* is evaluated will always be a chain of 4 frames:

  - A local frame for *h* linked to ...
  - A local frame for *g* linked to ...
  - A local frame for *f* linked to ...
  - The global frame.

- However, when there are multiple local frames for a particular function lying around, environment diagrams can help sort them out.

Do You Understand the Machinery? (VI)

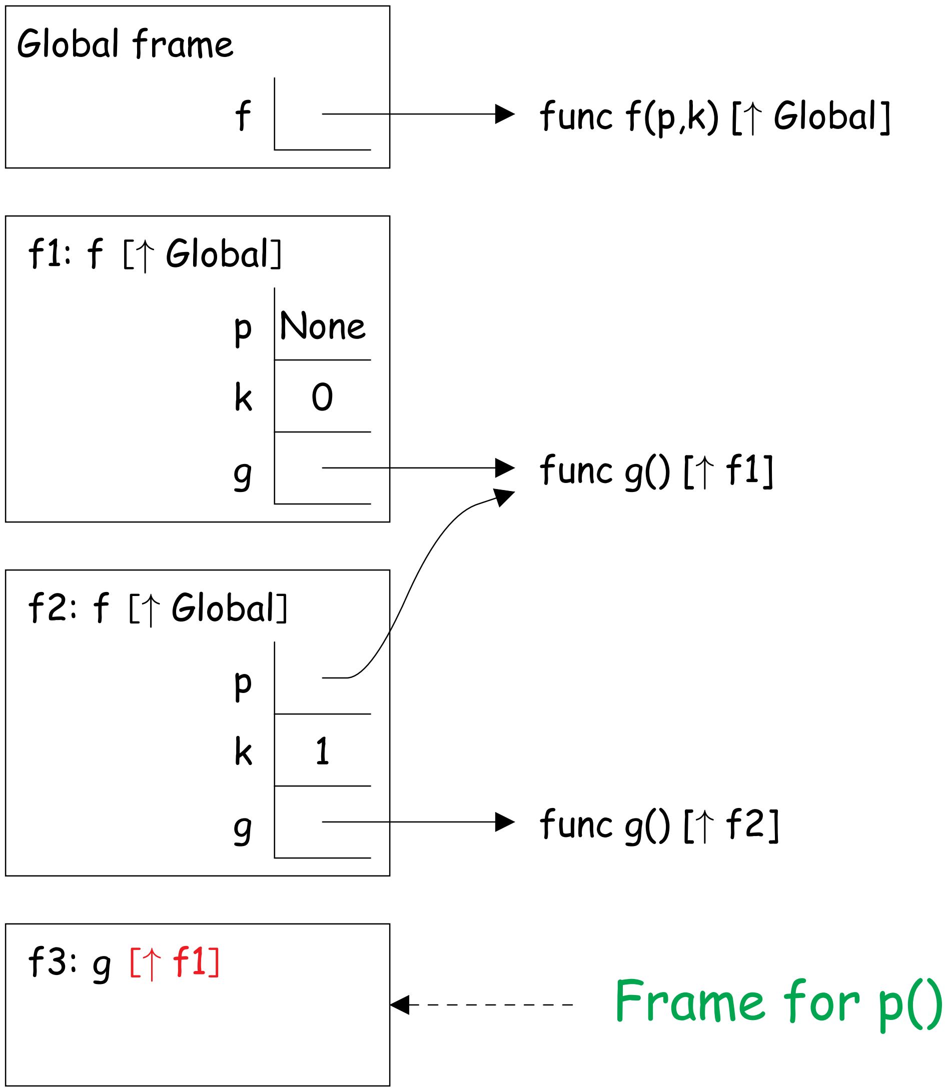
What is printed: (0, 1, or error) and why?

```
def f(p, k):
    def g():
        print(k)
    if k == 0:
        f(g, 1)
    else:
        p()
f(None, 0)
```

Answer (VI)

This prints 0. There are two local frames for f when p() is called (f1 and f2). The call to p() creates an instantiation of g whose parent is f1.

```
def f(p, k):
    def g():
        print(k)
    if k == 0:
        f(g, 1)
    else:
        p()
f(None, 0)
```



Global frame

f → func f(p,k) [↑ Global]

f1: f [↑ Global]

p None

k 0

g → func g() [↑ f1]

f2: f [↑ Global]

p → func g() [↑ f1]

k 1

g → func g() [↑ f2]

f3: g [↑ f1]

Frame for p()

Higher-Order Functions at Work: Iterative Update

- A general strategy for solving an equation:
  - Guess a solution
  - while your guess isn't good enough:
    - ∗ update your guess
- The three underlined segments are parameters to the process.
- The last two segments clearly require functions for their representation—a *predicate* function (returning true/false values), and a function from values to values.
- In code,

```
def iter_solve(guess, done, update):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result."""
```

Recursive Versions

```
def iter_solve(guess, done, update):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result."""
    if done(guess):
        return guess
    else:
        return iter_solve(update(guess), done, update)
```

or

```
def iter_solve(guess, done, update):
    def solution(guess):
        if done(guess):
            return guess
        else:
            return solution(update(guess))
    return solution(guess)
```

Iterative Version

```
def iter_solve(guess, done, update):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result."""
    while not done(guess):
        guess = update(guess)
    return guess
```

Adding a Safety Net

- In real life, we might want to make sure that the function doesn't just loop forever, getting no closer to a solution.

```
def iter_solve(guess, done, update, iteration_limit=32):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result.  Causes error if more than
    ITERATION_LIMIT applications of UPDATE are necessary."""
```

Adding a Safety Net: Code

- In real life, we might want to make sure that the function doesn't just loop forever, getting no closer to a solution.

```
def iter_solve(guess, done, update, iteration_limit=32):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result.  Causes error if more than
    ITERATION_LIMIT applications of UPDATE are necessary."""

    def solution(guess, iteration_limit):
        if done(guess):
            return guess
        elif iteration_limit <= 0
            raise ValueError("failed to converge")
        else:
            return solution(update(guess), iteration_limit-1)
    return solution(guess, iteration_limit)
```

Iterative Version

```
def iter_solve(guess, done, update, iteration_limit=32):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS, until DONE yields a true value
    when applied to the result.  Causes error if more than
    ITERATION_LIMIT applications of UPDATE are necessary."""

    while not done(guess):
        if iteration_limit <= 0:
            raise ValueError("failed to converge")
        guess, iteration_limit = update(guess), iteration_limit-1
    return guess
```

Using Iterative Solving For Newton's Method (I)

- Newton's method takes a function, its derivative, and an initial guess, and produces a result to some desired tolerance (that is, to some definition of "close enough").
- See `http://en.wikipedia.org/wiki/File:NewtonIteration_Ani.gif`
- Given a guess, x_k, compute the next guess, x_{k+1} by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

```
def newton_solve(func, deriv, start, tolerance):
    """Return x such that |FUNC(x)| < TOLERANCE, given initial
    estimate START and assuming DERIV is the derivatative of FUNC."""
    def close_enough(x):
        __________?__________
    def newton_update(x):
        __________?__________

    return iter_solve(start, close_enough, newton_update)
```

Using Iterative Solving for Newton's Method (II)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

```
def newton_solve(func, deriv, start, tolerance):
    """Return x such that |FUNC(x)| < TOLERANCE, given initial
    estimate START and assuming DERIV is the derivatative of FUNC."""
    def close_enough(x):
        return abs(func(x)) < tolerance
    def newton_update(x):
        return x - func(x) / deriv(x)

    return iter_solve(start, close_enough, newton_update)
```

Using newton_solve for $\sqrt{\cdot}$ and $\lg\cdot$

```
def square_root(a):
     return newton_solve(lambda x: x*x - a, lambda x: 2 * x,
                         a/2, 1e-5)

def logarithm(a, base = 2):
     return newton_solve(lambda x: base**x - a,
                         lambda x: x * base**(x-1),
                         1, 1e-5)
```

Dispensing With Derivatives

- What if we just want to work with a function, without knowing its derivative?
- Book uses an approximation:

```
def find_root(func, start=1, tolerance=1e-5):
    def approx_deriv(f, delta = 1e-5):
        return lambda x: (func(x + delta) - func(x)) / delta
    return newton_solve(func, approx_deriv(func), start, tolerance)
```

- This is nice enough, but looks a little ad hoc (how did I pick delta?).
- Another alternative is the *secant method.*

The Secant Method

- Newton's method was

$$x_{k+1} = x_k - \frac{f(x)}{f'(x)}$$

- The secant method uses that last two values to get (in effect) a replacement for the derivative:

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

- See `http://en.wikipedia.org/wiki/File:Secant_method.svg`
- But this is a problem for us: so far, we've only fed the update function the value of x_k each time. Here we also need x_{k-1}.
- How do we generalize to allow arbitrary extra data (not just x_{k-1})?

Generalized iter_solve

```
def iter_solve2(guess, done, update, state=None):
    """Return the result of repeatedly applying UPDATE,
    starting at GUESS and STATE, until DONE yields a true value
    when applied to the result.  Besides a guess, UPDATE
    also takes and returns a state value, which is also passed to
    DONE."""
    while not done(guess, state):
        guess, state = update(guess, state)
    return guess
```

Using Generalized iter_solve2 for the Secant Method

The secant method:

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

```
def secant_solve(func, start0, start1, tolerance):
    def close_enough(x, state):
        return abs(func(x)) < tolerance
    def secant_update(xk, xk1):
        return (xk - func(xk) * (xk - xk1)
                                / (func(xk) - func(xk1),
                xk)
    return iter_solve2(start1, close_enough, secant_update, start0)
```