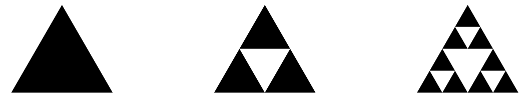
Lecture #7: Tree Recursion

Announcements:

- Hog contest to be released today (I think). It is optional.
- Also watch for HW #2.

Subproblems and Self-Similarity

- Recursive routines arise when solving a problem naturally involves solving smaller instances of the same problem.
- A classic example where the subproblems are visible is *Sierpinski's Triangle* (aka bit Sierpinski's Gasket).
- This triangle may be formed by repeatedly replacing a figure, initially a solid triangle, with three quarter-sized images of itself (1/2 size in each dimension), arranged in a triangle:



- Or we can think creating a "triangle of order N and size S" by drawing either
  - a solid triangle with side S if $N = 0$, or
  - three triangles of size $S/2$ and order $N-1$ arranged in a triangle.

The Gasket in Python

- We can describe this as a recursive Python program that produces Postscript output.

```
sin60 = sqrt(3) / 2
def make_gasket(x, y, s, n, output):
    """Write Postscript code for a Sierpinski's gasket of order N
    with lower-left corner at (X, Y) and side S on OUTPUT."""
    if n == 0:
        draw_solid_triangle(x, y, s, output)
    else:
        make_gasket(x, y, s/2, n - 1, output)
        make_gasket(x + s/2, y, s/2, n - 1, output)
        make_gasket(x + s/4, y + sin60*s/2, s/2, n - 1, output)

def draw_solid_triangle(x, y, s, output):
    "Draw a solid triangle lower-left corner at (X, Y) and side S."
    print("{0} {1} moveto "   # Go x, y
          "{2} 0 rlineto "    # Horizontal move by s units
          "-{3} {4} rlineto " # Move up and to left
          "closepath fill"    # Close path and fill with black
          .format(x, y, s, s/2, s*sin60), file=output)
```


Aside: Using the Functions

- Just to complete the picture, we can use `make_gasket` to create a standalone Postscript file on a given file.

```
def draw_gasket(n, output=sys.stdout):
    print("%!", file=output)
    make_gasket(100, 100, 400, 8, output)
    print("showpage", file=output)
```

- And just for fun, here's some Python magic to display triangles automatically (uses gs, the Ghostscript interpreter for Postscript).

```
from subprocess import Popen, PIPE, DEVNULL

def make_displayer():
    """Create a Ghostscript process that displays its input (sent in through
    .stdin)."""
    return Popen("gs", stdin=PIPE, stdout=DEVNULL)

>>> d = make_displayer()
>>> draw_gasket(5, d.stdin)
>>> draw_gasket(10, d.stdin)
```

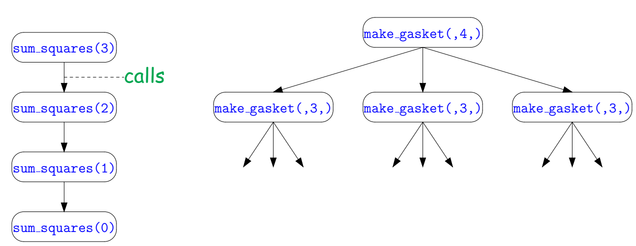

Aside: The Gasket in Pure Postscript

- One can also perform the logic to generate figures in Postscript directly, which is itself a full-fledged programming language:

```
%!

/sin60 3 sqrt 2 div def

/make_gasket {
    dup 0 eq {
        3 index 3 index moveto 1 index 0 rlineto 0 2 index rlineto
                1 index neg 0 rlineto closepath fill
    } {
        3 index 3 index 3 index 0.5 mul 3 index 1 sub make_gasket
        3 index 2 index 0.5 mul add 3 index 3 index 0.5 mul
              3 index 1 sub make_gasket
        3 index 2 index 0.25 mul add 3 index 3 index 0.5 mul add
              3 index 0.5 mul 3 index 1 sub make_gasket
    } ifelse
    pop pop pop pop
} def

100 100 400 8 make_gasket showpage
```

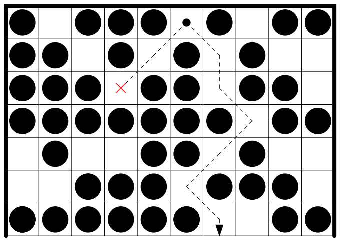

Tree Recursion

- The `make_gasket` function is an example of a *tree recursion*, where each call makes multiple recursive calls on itself.
- A linear recursion such as that on the left (for `sum_sqaures`) produces a pattern of calls such as that on the left, while `make_gasket` produces the pattern on the right—an instance of what we call a *tree* in computer science.



Finding a Path

- Consider the problem of finding your way through a maze of blocks:

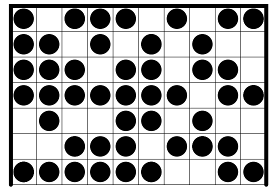


- From a given starting square, one can move down one level and up to one column left or right on each step, as long as the square moved to is unoccupied.
- Problem is to find a path to the bottom layer.
- Diagram shows one path that runs into a dead end and one that escapes.

Path-Finding Program

- Translating the problem into a function specification:

```
def is_path(blocked, x0, y0):
   """True iff there is a path of squares from (X0, Y0) to some
   square (x1, 0) such that all squares on the path (including ends)
   are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied. Each step of a
   path goes down one row and 1 or 0 columns left or right."""
```



This grid would be represented by a predicate M where, e.g, `M(0,0)`, `M(1,0)`, `M(1,2)`, `not M(1, 1)`, `not M(2,2)`.

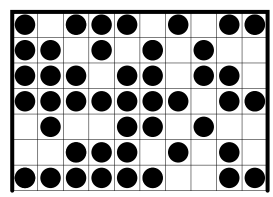
Here, `is_path(M, 5, 6)` is true; `is_path(M, 1, 6)` and `is_path(M, 6, 6)` are false.

is_path Solution

```
def is_path(blocked, x0, y0):
   """True iff there is a path of squares from (X0, Y0) to some
   square (x1, 0) such that all squares on the path (including ends)
   are unoccupied.  BLOCKED is a predicate such that BLOCKED(x, y)
   is true iff the grid square at (x, y) is occupied. Each step of a
   path goes down one row and 1 or 0 columns left or right."""


   if ____________:

       return ____________
   elif ____________:

       return ____________
   else:

       return ________________________________________________
```


Variation I

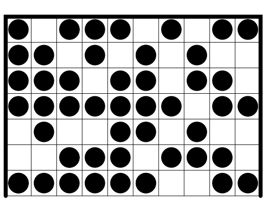
```
def num_paths(blocked, x0, y0):
   """Return the number of paths that run from
   (X0, Y0) to some unoccupied square (x1, 0).
   BLOCKED is a predicate such that BLOCKED(x, y) is
   true iff the grid square at (x, y) is occupied. """
```



Result of `num_paths(M, 5, 6)` is 1 (original `M`) If `M2` is the maze above (missing (7, 1)), then result of `num_paths(M2, 5, 6)` is 5.

Variation II

```
def find_path(blocked, x0, y0):
   """Return a string containing the steps in a path
   from (X0, Y0) to some unoccupied square (x1, 0),
   or None if not is_path(BLOCKED, X0, Y0). BLOCKED is a
   predicate such that BLOCKED(x, y) is true iff the
   grid square at (x, y) is occupied. """
```



Possible result of `find_path(M, 5, 6)`:

```
"(5, 6) (6, 5) (6, 4) (7, 3) (6, 2) (5, 1) (6, 0)"
```


find_path Solution

```
def find_path(blocked, x0, y0):
   """Return a string containing the steps in a path
   from (X0, Y0) tosome unoccupied square (x1, 0),
   or None if not is_path(BLOCKED, X0, Y0). BLOCKED is a
    predicate such that BLOCKED(x, y) is true iff the
   grid square at (x, y) is occupied. """
```

A Change in Problem

- Suppose we changed the definition of "path" for the maze problems to allow paths to go left or right without going down.
- And suppose we changed solutions in the obvious way, adding clauses for the $(x_0 - 1, y_0)$ and $(x_0 + 1, y_0)$ cases.
- Will this work? What would happen?

And a Little Analysis

- All our linear recursions took time proportional (in some sense) to the size of the problem.
- What about `is_path`?