Lecture #19: Complexity and Orders of Growth

Annoucements

- Sign up for alternative test #2 times within next two weeks
  `http://goo.gl/forms/VhI4rHw3LW`.
- Hog Contest Results Up!
  `http://cs61a.org/proj/hog_contest/results/`.

Complexity

- Certain problems take longer than others to solve, or require more storage space to hold intermediate results.
- We refer to the *time complexity* or *space complexity* of a problem.
- But what does it mean to say that a certain *program* has a particular complexity?
- What does it mean for an *algorithm*?
- What does it mean for a *problem*?

A Direct Approach

- Well, if you want to know how fast something is, you can time it.
- Python happens to make this easy:

```
>>> def fib(n):
...     if n <= 1: return n
...     else: return fib(n-2) + fib(n-1)
...
>>> from timeit import repeat
>>> repeat('fib(10)', 'from __main__ import fib', number=5)
[0.000491..., 0.000486..., 0.000487...]
>>> repeat('fib(20)', 'from __main__ import fib', number=5)
[0.060..., 0.060..., 0.060...]
>>> repeat('fib(30)', 'from __main__ import fib', number=5)
[7.74..., 7.81..., 7.81...]
```

- `repeat(Stmt, Setup, number=N)` says

  Execute Setup (a string containing Python code), then execute Stmt (a string) N times. Repeat this process 3 times and report the time required for each repetition.

A Direct Approach, Continued

- You can also use this from the command line:

```
...# python3 -m timeit --setup='from fib import fib' 'fib(10)'
10000 loops, best of 3: 97 usec per loop
```

- This command automatically chooses a number of executions of fib to give a total time that is large enough for an accurate average, repeats 3 times, and reports the best time.

Strengths and Problems with Direct Approach

- Good: Gives actual times; answers question completely for given input and machine.
- Bad: Results apply only to tested inputs.
- Bad: Results apply only to particular programs and platforms.
- Bad: Cannot tell us anything about complexity of algorithm or of problem.

But Can't We Extrapolate?

- Why not try a succession of times, and use that to figure out timing in general?

```
...# for t in 5 10 15 20 25 30; do
>     echo -n "$t: "
>     python3 -m timeit --setup='from fib import fib' "fib($t)"
> done
5: 100000 loops, best of 3: 8.16 usec per loop
10: 10000 loops, best of 3: 96.8 usec per loop
15: 1000 loops, best of 3: 1.08 msec per loop
20: 100 loops, best of 3: 12 msec per loop
25: 10 loops, best of 3: 133 msec per loop
30: 10 loops, best of 3: 1.47 sec per loop
```

- This looks to be exponential in t with exponent of ≈ 1.6.
- But... what if the program special-cases some inputs?
- ...and this still only works for a particular program and machine.

Worst Case, Average Case

- To avoid the problem of getting results only for particular inputs, we usually ask a more general question, such as:
  - What is the *worst case* time to compute f(X) as a function of the size of X, or
  - what is the *average case* time to compute f(X) over all values of X (weighted by likelihood).
- Average case is hard, so we'll let other courses deal with it.
- But now we seem to have a harder problem than before: how do we get worst-case times? Doesn't that require testing all cases?
- And when we do, aren't we still sensitive to machine model, compiler, etc.?

Example: Linear Search

- Consider the following search function:

```
def near(L, x, delta):
    """True iff X differs from some member of sequence L by no
    more than DELTA."""
    for y in L:
        if abs(x-y) <= delta:
            return True
    return False
```

- There's a lot here we don't know:
  - How long is sequence `L`?
  - Where in `L` is `x` (if it is)?
  - What kind of numbers are in `L` and how long do they take to compare?
  - How long do `abs` and subtract take?
  - How long does it take to create an iterator for `L` and how long does its `__next__` operation take?
- So what can we meaningfully say about complexity of `near`?

What to Measure?

- If we want general answers, we have to introduce some "strategic vagueness."
- Instead of looking at times, we can consider number of "operations." Which?
- The total time consists of
  1. Some fixed overhead to start the function and begin the loop.
  2. Per-iteration costs: subtraction, `abs`, `__next__`, `<=`
  3. Some cost to end the loop.
  4. Some cost to return.
- So we can collect total operations into one "fixed-cost operation" (items 1, 3, 4), plus M(L) "loop operations" (item 2), where $M(L)$ is the number of items in `L` up to and including the `y` that comes within `delta` of `x` (or the length of `L` if no match).

What Does an "Operation" Cost?

- But these "operations" are of different kinds and complexities, so what do we really know?
- Assuming that each operation represents some range of possible minimum and maximum values (constants), we can say that

$$
\begin{array}{l}
\quad \textit{min-fixed-cost} + M(\mathsf{L}) \times \textit{min-loop-cost} \\
\leq \\
\quad C_{\text{near}}(L) \\
\leq \\
\quad \textit{max-fixed-cost} + M(\mathsf{L}) \times \textit{max-loop-cost}
\end{array}
$$

  where $C_{\text{near}}(L)$ is the cost of `near` on a list where the program has to look at $M(L)$ items.
- In the worst case M(L) $==$ len(L) and in the best, M(L) ≤ 1, so

  min-fixed-cost $\leq C_{\text{near}}(L) \leq$ *max-fixed-cost*$+$len(L)$)\times$*max-loop-cost*.
- Simpler, but still clumsy, and the numbers are not going to be precise anyway. Would be nice to have a cleaner notation.

Operation Counts and Scaling

- Instead of getting precise answers in units of physical time, we therefore settle for a proxy measure that will remain meaningful over changes in architecture or compiler.
- Choose some operation(s) of interest and count how many times they occur.
- Examples:
  - How many times does fib get called recursively during computation of fib(N)?
  - How many addition operations get performed by fib(N)?
- You can no longer get precise times, but if the operations are well-chosen, results are *proportional* to actual time for different values of N.
- Thus, we look at how computation time *scales* in the worst case.
- Can compare programs/algorithms on the basis of which scale better.

Some Intuition on Meaning of Growth

- How big a problem can you solve in a *given* time?
- In the following table, left column shows time in microseconds to solve a *given* problem as a function of problem size N (assuming perfect scaling and that problem size 1 takes 1μsec).
- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.
- $N =$ problem size

Time (μsec) for problem size N	**Max N Possible in** 1 second	1 hour	1 month	1 century
$\lg N$	10^{300000}	$10^{1000000000}$	$10^{8 \cdot 10^{11}}$	$10^{9 \cdot 10^{14}}$
N	10^6	$3.6 \cdot 10^9$	$2.7 \cdot 10^{12}$	$3.2 \cdot 10^{15}$
$N \lg N$	63000	$1.3 \cdot 10^8$	$7.4 \cdot 10^{10}$	$6.9 \cdot 10^{13}$
N^2	1000	60000	$1.6 \cdot 10^6$	$5.6 \cdot 10^7$
N^3	100	1500	14000	150000
2^N	20	32	41	51

Asymptotic Results

- Sometimes, results for "small" values are not indicative.
- E.g., suppose we have a prime-number tester that contains a look-up table of the primes up to 1,000,000,000 (about 50 million primes).
- Tests for numbers up to 1 billion will be faster than for larger numbers.
- So in general, we tend to ask about *asymptotic* behavior of programs: as size of input goes to infinity.

Expressing Approximation

- So, we are looking for measures of program performance that give us a sense of how computation time scales with size of input.
- And we are further interested in ignoring finite sets of special cases that a given program can compute quickly.
- Finally, precise worst-case functions can be very complicated, and the precision is generally not terribly important anyway.
- These considerations motivate the use of *order notation* to express approximations of execution time or space.

The Notation

- Suppose that f is a function of one parameter returning real numbers.
- We use the notation $O(f)$ to mean "the set of all one-parameter functions whose absolute values are eventually bounded above by some multiple of f's absolute value." Formally:

  $$O(f) = \{g \mid |g(x)| \leq p \cdot |f(x)| \text{ when } x > M, \text{for some constants } p, M\}$$

- So we can write $g \in O(f)$ to mean "whenever n is large enough, $|g(n)| \leq p \cdot |f(n)|$ for some constant p."
- We define

  $$f \in \Omega(g) \stackrel{\text{def}}{=} g \in O(f).$$

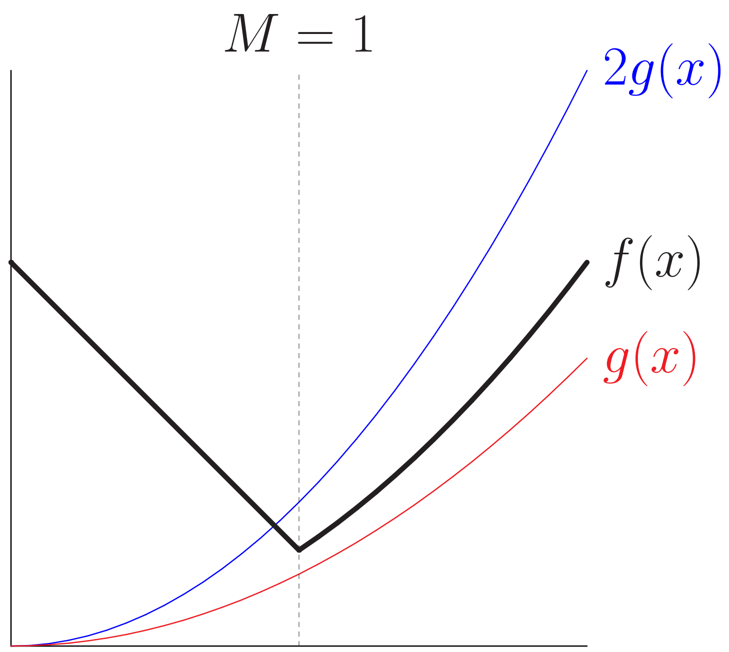
  or "f is eventually bounded below by a multiple of g."

- And finally those bounded both above and below:

  $$\begin{aligned}\Theta(f) &= \Omega(f) \cap O(f) \\ &= \{g \mid p_1 \cdot |f(x)| \leq p_2 \cdot |f(x)| \text{ when } x > M\}\end{aligned}$$

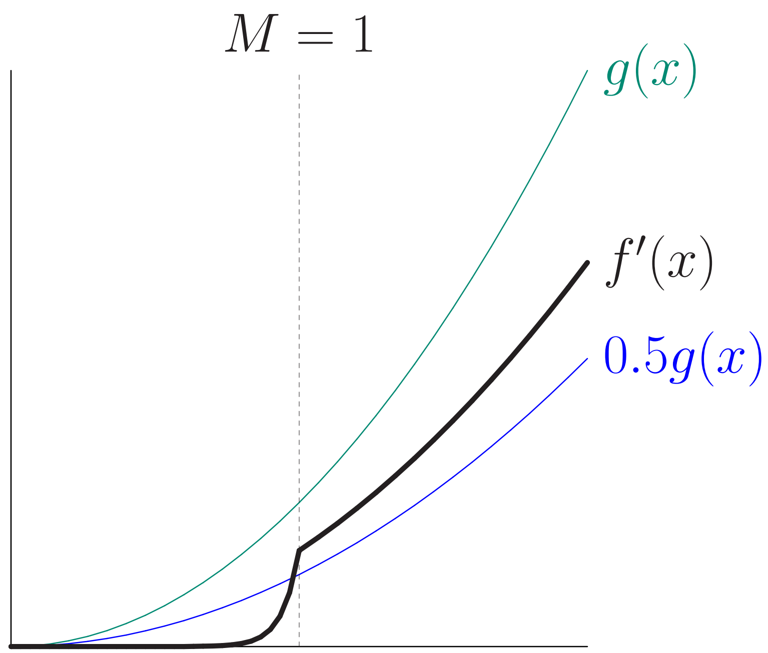
  for some constants $0 < p_1 \leq p_2$, and M.

Illustration



- Here, $f \in O(g)$ ($p = 2$, see blue line), even though $f(x) > g(x)$. Likewise, $f \in \Omega(g)$ ($p = 1$, see red line), and therefore $f \in \Theta(g)$.
- That is, $f(x)$ is eventually (for $x > M = 1$) no more than proportional to $g(x)$ and no less than proportional to $g(x)$.

Illustration, contd.



- Here, $f' \in \Omega(g)$ ($p = 0.5$), even though $g(x) > f'(x)$ everywhere.

Notational Quirks

- You may have seen $O(\cdot)$ notation in math, where we say things like

$$f(x) \in f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(f'''(0)x^3)$$

- Adding or multiplying sets of functions produces sets of functions. Thus, $x^2 + O(g)$ means "the set of functions of x returning $x^2 + h(x)$, where $h \in O(g)$.
- Well, to be picky, we really ought to write

$$f \in (\lambda x.f(0) + f'(0)x + \ldots) + O(\lambda x.f'''(0)x^3)$$

  but that really gets really tedious...
- ...so if $E(x)$ is some expression involving x, we usually abbreviate $\lambda x.E(x)$ as just $E(x)$. Example: $n + 1 \in O(n^2)$
- I prefer $\in$ to the traditional $f(x) = f(0) + \cdots$, since the latter makes no formal sense (the left side is a function and the right is a set of functions.)
- Finally, we will sometimes write $f \in \Theta(g)$ even when f and g are functions of something non-numeric (like lists).

Using Asymptotic Estimates

- Going back to linear search,

$$\textit{min-fixed-cost} + M(L) \times \textit{min-loop-cost} \leq C_{\rm near}(L)$$
$$\leq \textit{max-fixed-cost} + M(L) \times \textit{max-loop-cost}$$

- Claim: we can state this more cleanly as $C_{\rm near}(L) \in O(M(L))$ and $C_{\rm near}(L) \in \Omega(M(L))$, or even more concisely: $C_{\rm near}(L) \in \Theta(M(L))$.
- Why? $C_{\rm near}(M(L)) \in O(M(L))$ if $C_{\rm near}(M(L)) \leq K \cdot M(L)$ for sufficiently large $M(L)$, by definition.
- And if if K_1 and K_2 are any (non-negative) constants, then $K_1 + K_2 \cdot M(L) \leq (K_1 + K_2) \cdot M(L)$ for $M(L) > 1$.
- Likewise, $K_1 + K_2 \cdot M(L) \geq K_2 \cdot M(L)$ for $M > 0$.
- And we can go even farther. If the sequence, `L`, has length $N(L)$, then we know that $M(L) \leq N(L)$. Therefore, we can say $C_{\rm near}(L) \in O(N(L))$.
- Is $C_{\rm near}(L) \in \Omega(N(L))$?

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$$\textit{min-fixed-cost} + M(L) \times \textit{min-loop-cost} \leq C_{\text{near}}(L)$$
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- And we can go even farther. If the sequence, `L`, has length $N(L)$, then we know that $M(L) \leq N(L)$. Therefore, we can say $C_{\text{near}}(L) \in O(N(L))$.
- Is $C_{\text{near}}(L) \in \Omega(N(L))$? No: can only say $C_{\text{near}}(L) \in \Omega(1)$.

Best/Worst Cases

- We can simplify still further by not trying to give results for particular inputs, but instead giving summary results for *all inputs of the same "size."*
- Here, "size" depends on the problem: could be magnitude, length (of list), cardinality (of set), etc.
- Since we don't consider specific inputs, we have to be less precise.
- Typically, the figure of interest is the *worst case over all inputs of the same size.*
- Also makes sense to talk about the *best case* over all inputs of the same size, or the *average case* over all inputs of the same size (weighted by likelihood). These are rarer, though.
- From preceding discussion, since $C_{\text{near}}(N(L)) \in O(N(L))$, it follows that $C_{\text{wc}}(N) \in O(N)$, where $C_{\text{wc}}(N)$ is "worst-case cost of `near` over all lists of size N."

Best of the Worst

- We just saw that $C_{\text{wc}}(N) \in O(N)$.
- But in addition, it's also clear that $C_{\text{wc}}(N) \in \Omega(N)$.
- So we can say, most concisely, $C_{\text{wc}}(N) \in \Theta(N)$.
- Generally, when a worst-case time is not $\Theta(\cdot)$, it indicates either that
  - We don't know (haven't proved) what the worst case really is, just put limits on it,
    - Most often happens when we talk about the worst-case for a *problem:* "what's the worst case for the best possible algorithm?"
  - Or we know what the worst-case time is, but it's messy, so we settle for approximations that are easier to deal with.