Lecture 5: Higher-Order Functions

Brian Hou
June 27, 2016

Announcements

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  - Environment Diagrams and Higher-Order Functions

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  - Environment Diagrams and Higher-Order Functions
- Group Tutoring is available! See Piazza for details

Roadmap

Introduction

Functions

Data

Mutability

Objects

Interpretation

Paradigms

Applications

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Objects

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Applications

- This week (Functions), the goals are:
  - To understand the idea of *functional abstraction*
  - To study this idea through:
    - higher-order functions
    - recursion
    - orders of growth

Higher-Order Functions

Generalizing Computations

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$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \qquad = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \qquad = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \qquad = 3.04$$

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Generalizing Computations (demo)

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Generalizing Computations

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def sum_naturals(n):
    total, k = 0, 1
    while k <= n:
        total, k = total + k, k + 1
    return total

def sum_cubes(n):
    total, k = 0, 1
    while k <= n:
        total, k = total + pow(k, 3), k + 1
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Summation Example

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cube = lambda k: pow(k, 3)

def summation(n, term):
    """Sum the first N terms of a sequence.

    >>> summation(5, cube)
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Function of a single argument (*not called* "*term*")

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A parameter that will be bound to a function

The cube function is passed as an argument value

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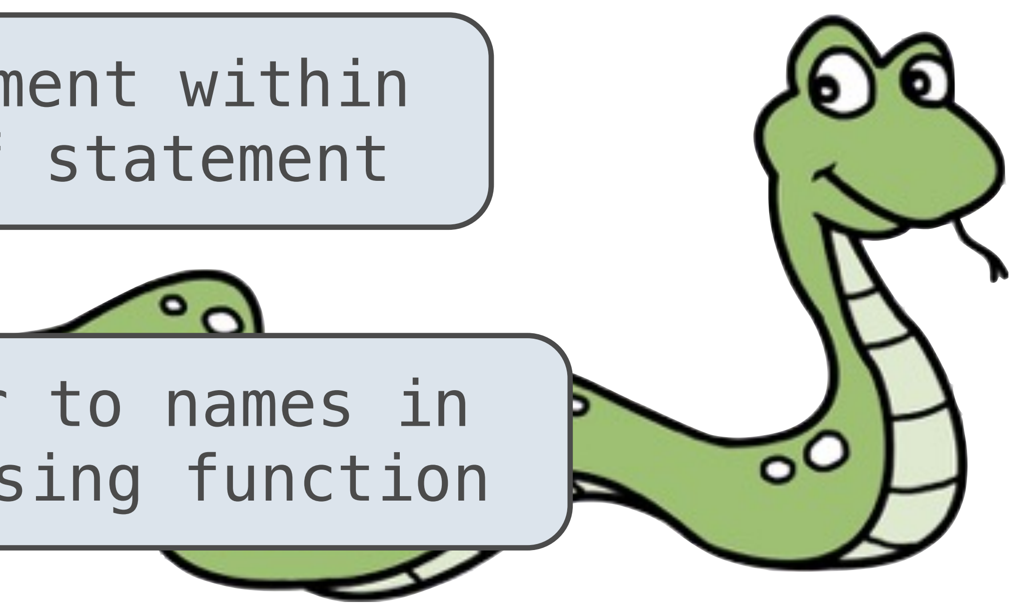

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- Remove repetition from programs
- Separate concerns among functions

Break!

Environments (Round 2)

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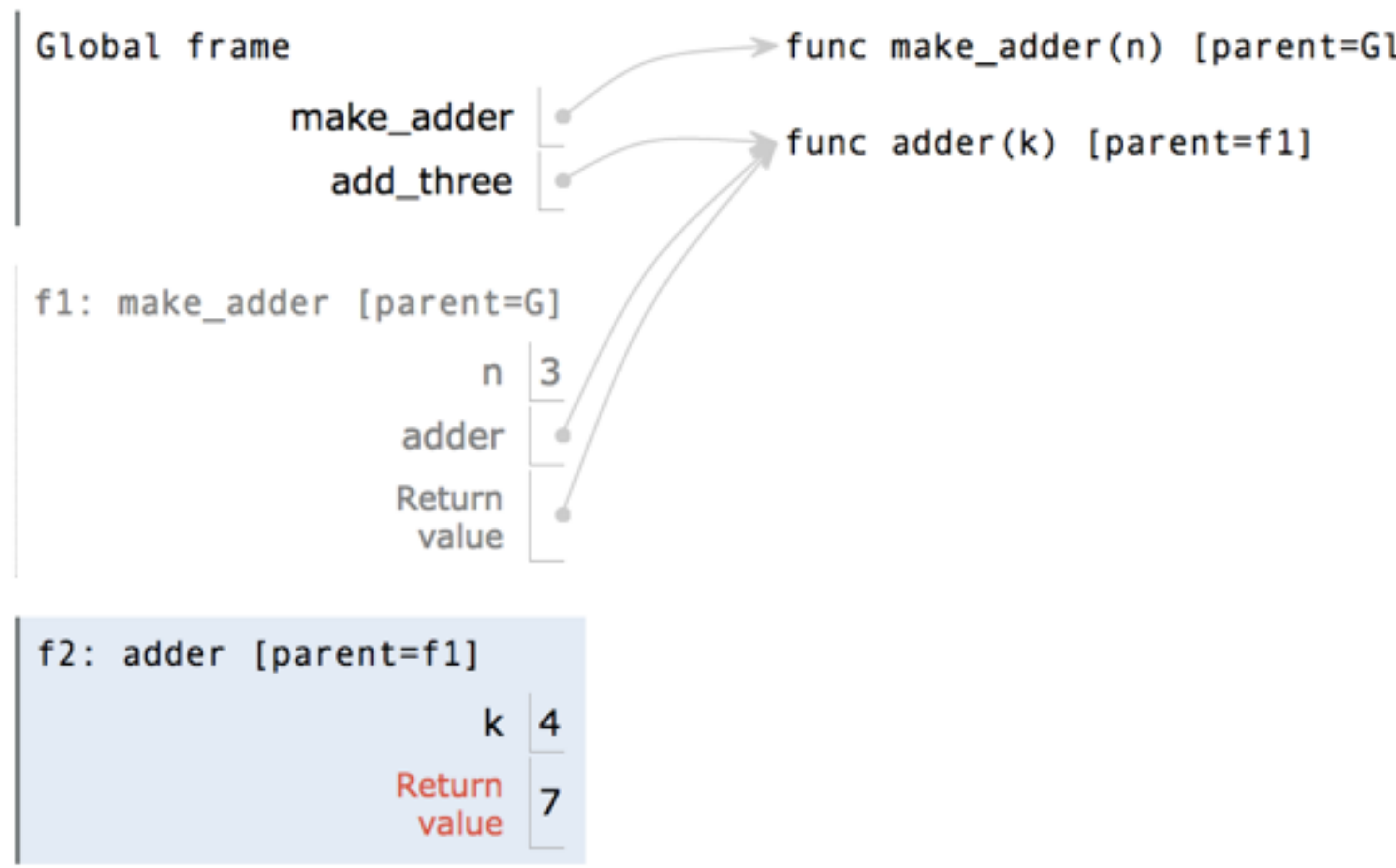

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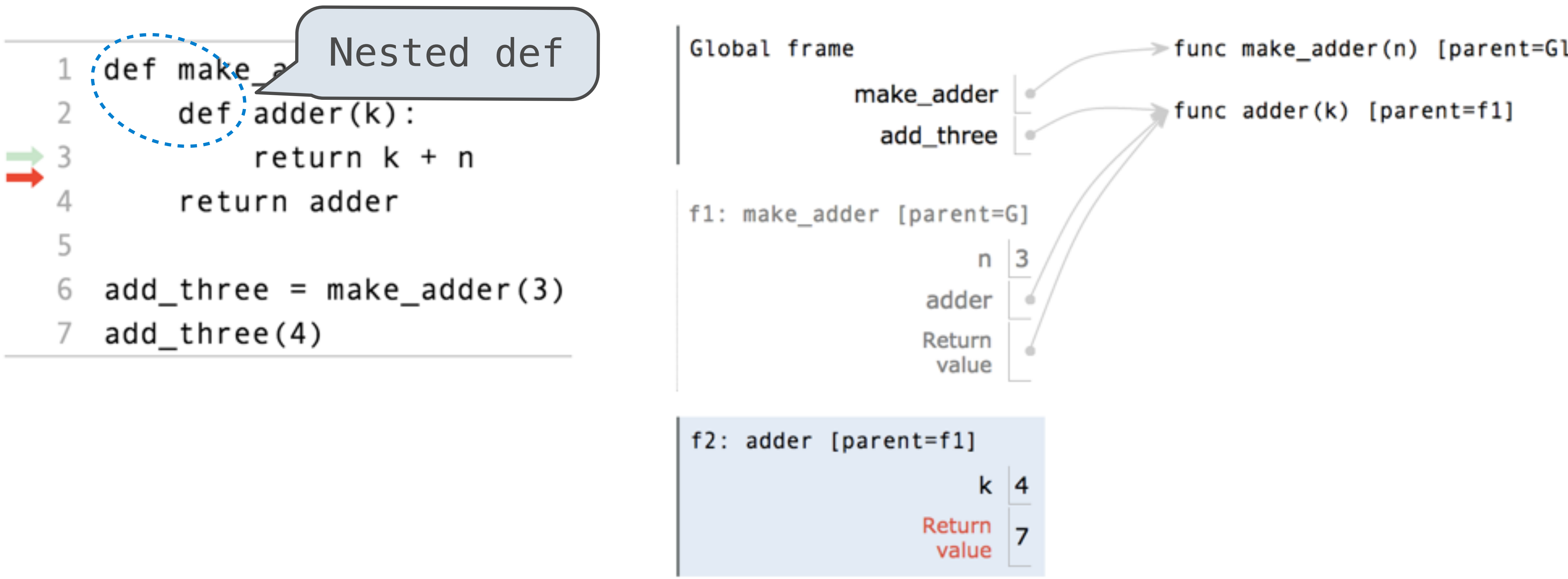

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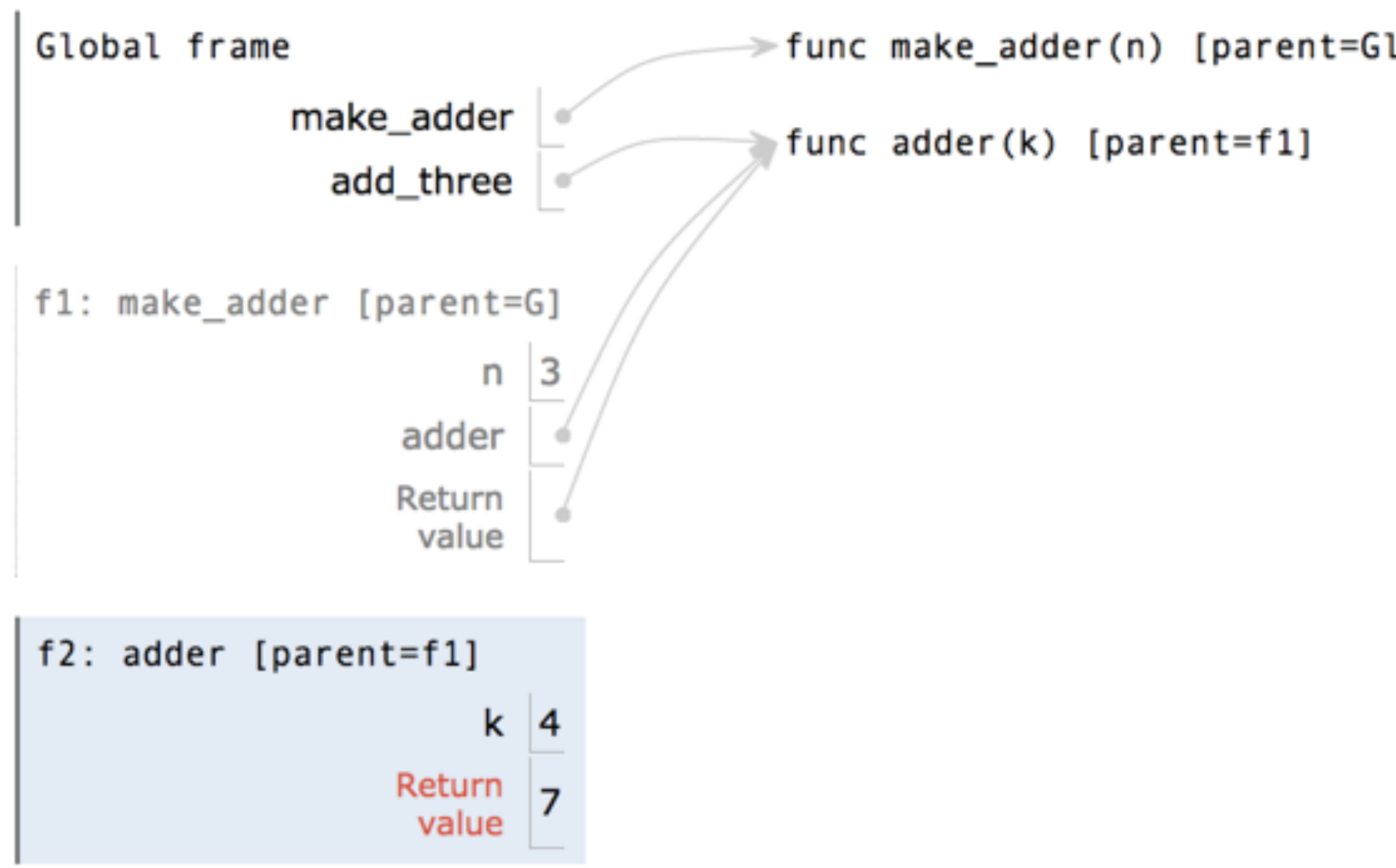

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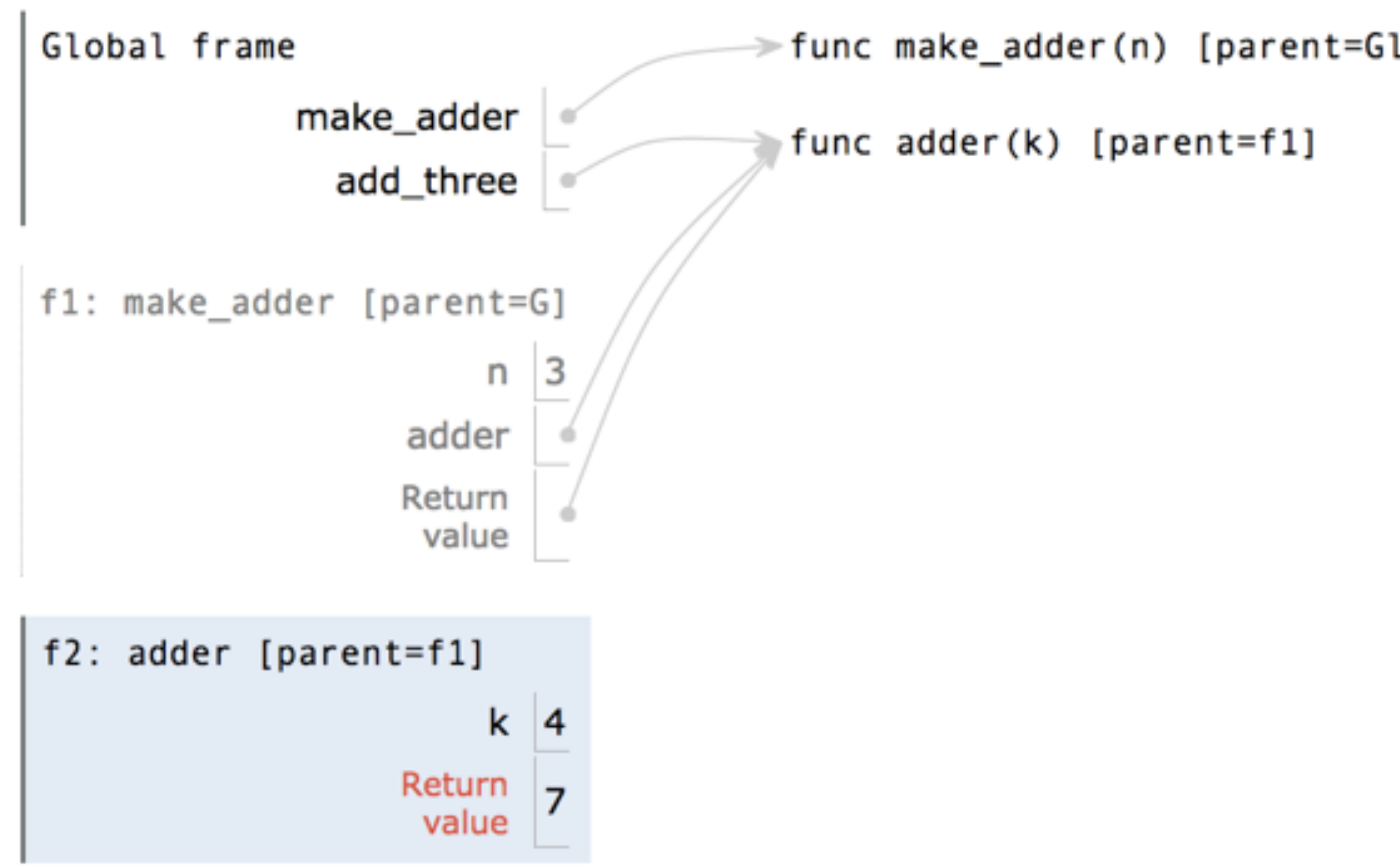

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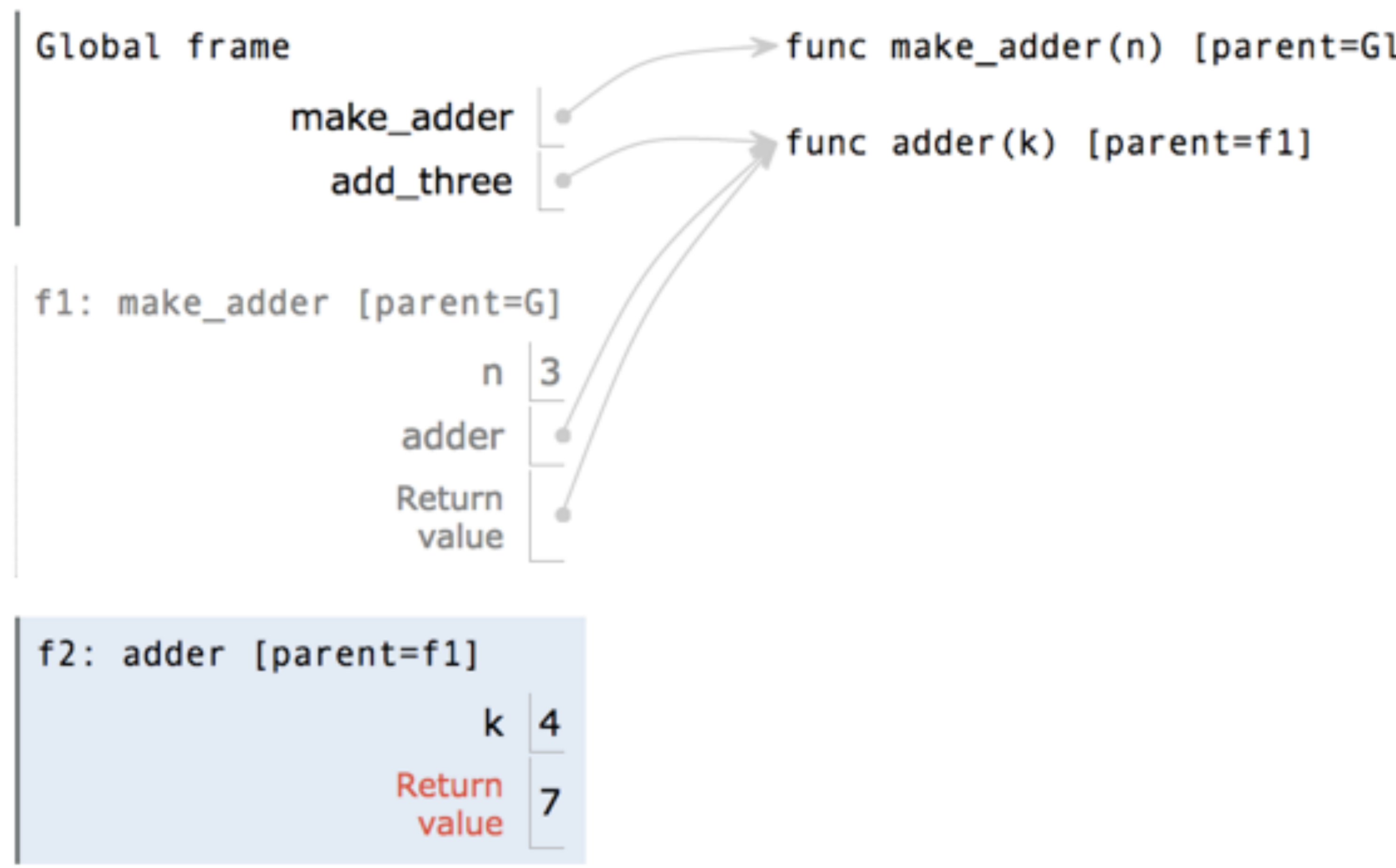

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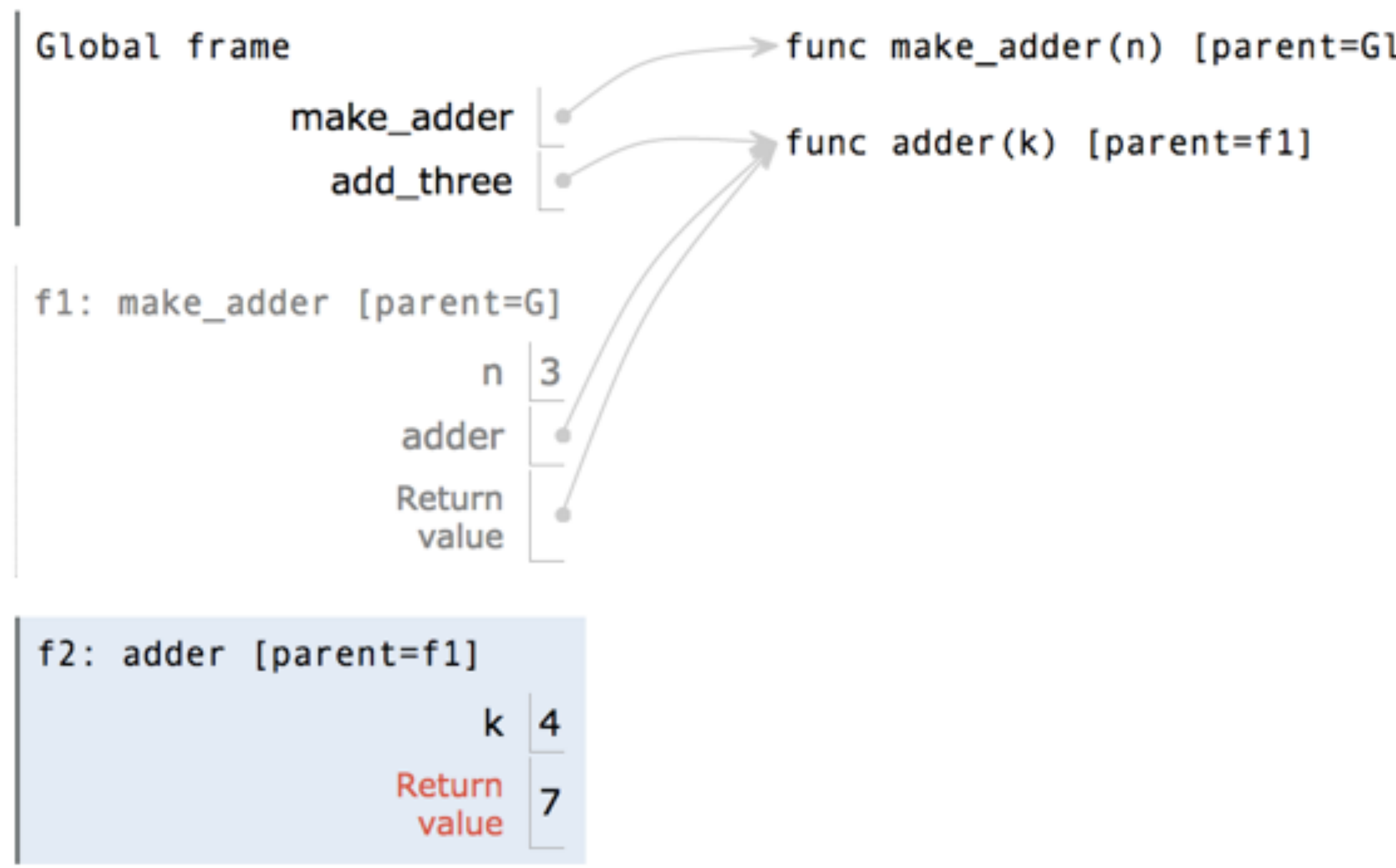

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Rules for def Statements:

1. Create a function with signature <name>(<parameters>) **and parent [parent=<label>]** (parent is the current frame)

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Rules for calling user-defined functions:

1. Create a new environment frame
2. **Copy the parent of the function to the local frame: [parent=<label>]**
3. Bind the function's parameters to its arguments in that frame
4. Execute the body of the function in the new environment

Function Composition

Environment Diagram

```
def square(x):
    return x * x

def make_adder(n):
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def compose1(f, g):
    def h(x):
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compose1(square, make_adder(2))(3)
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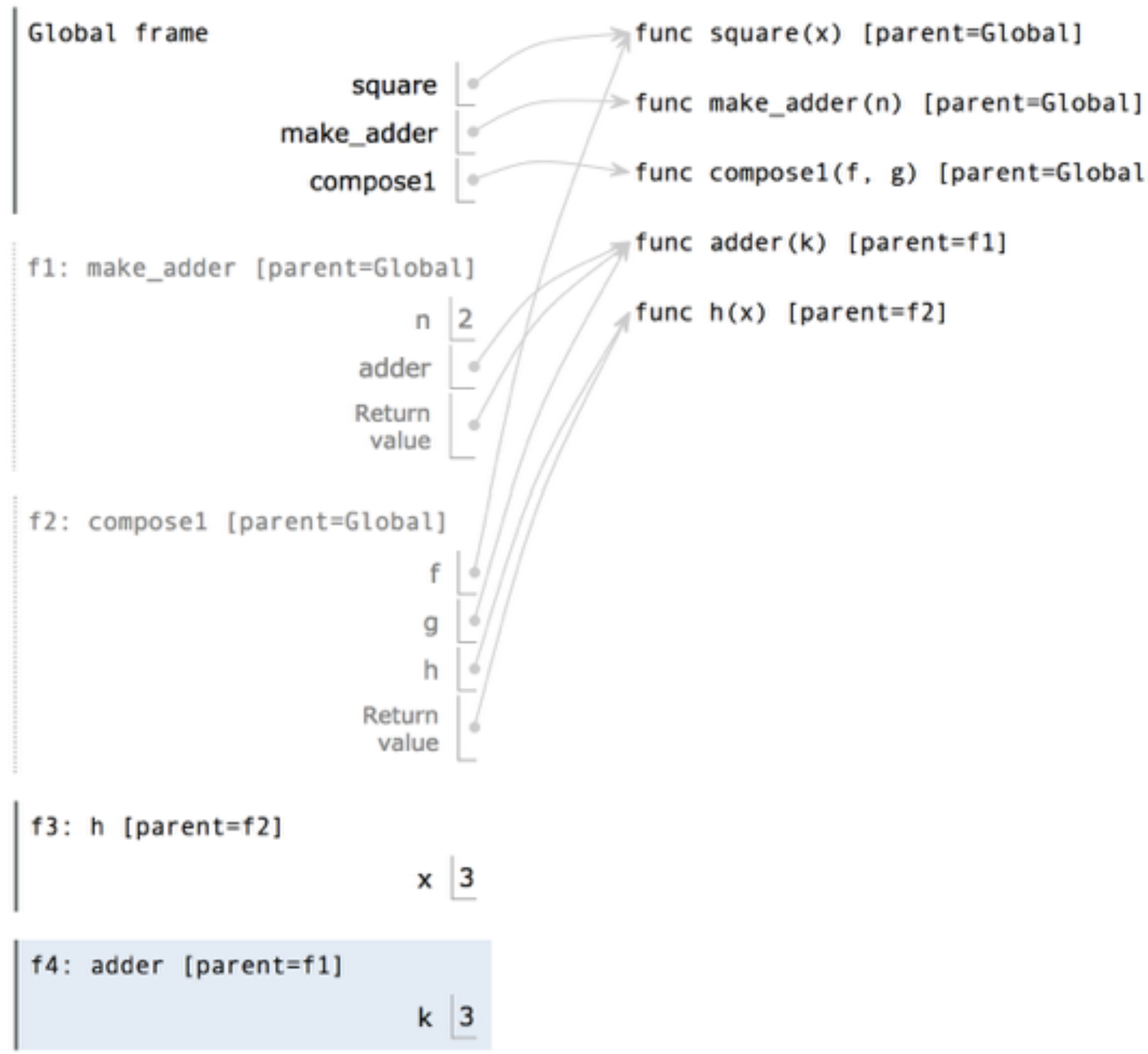

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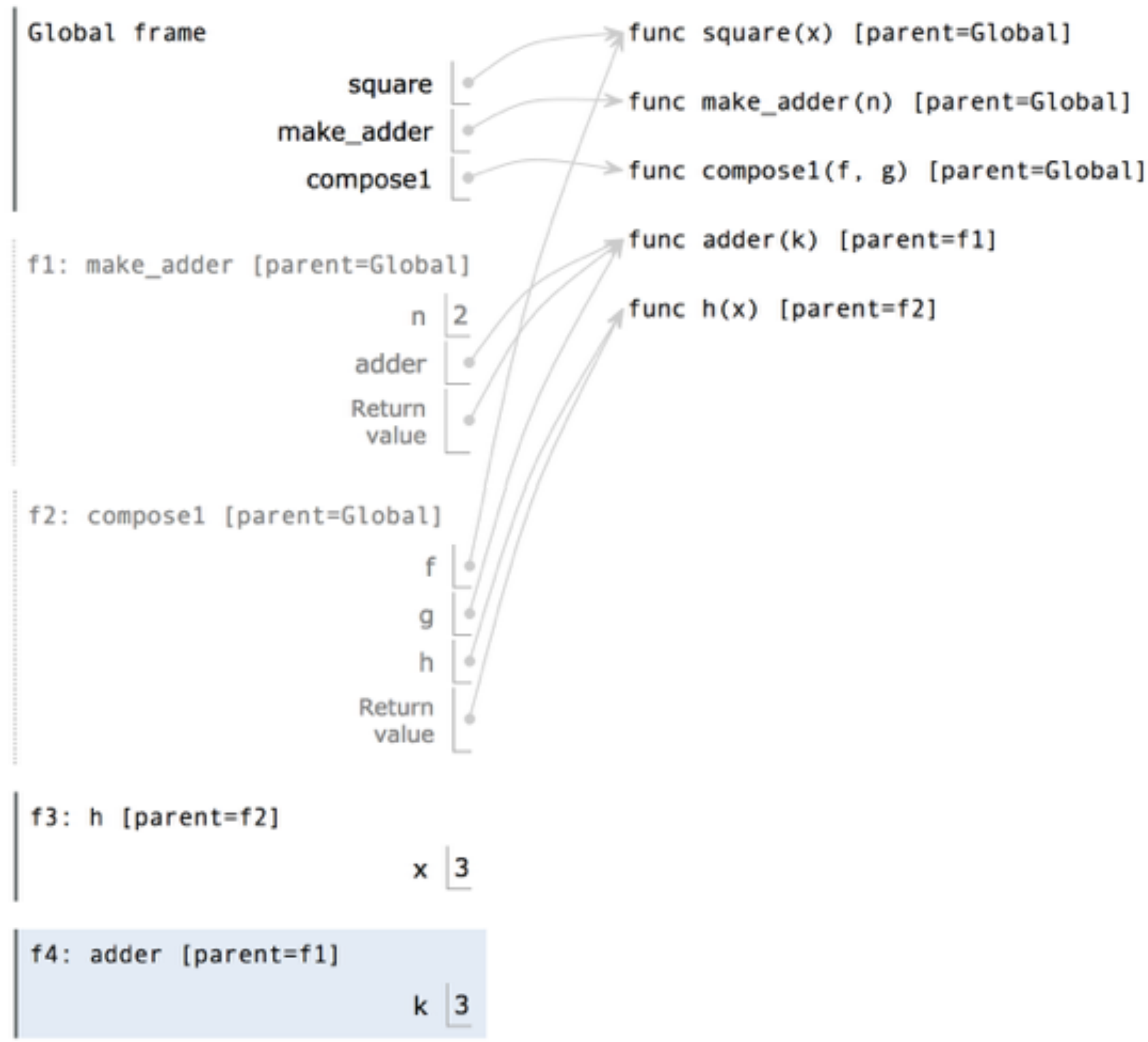

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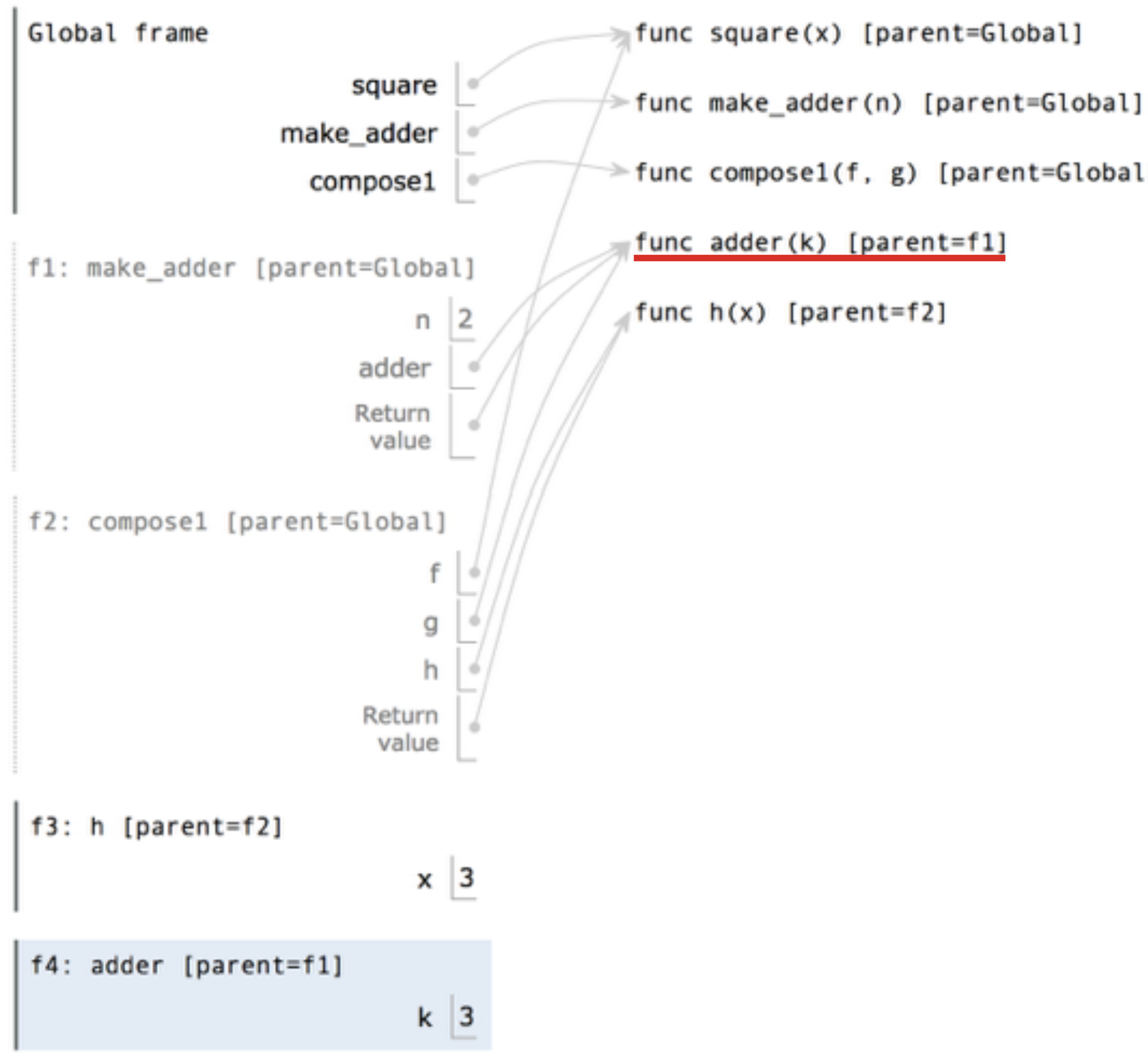

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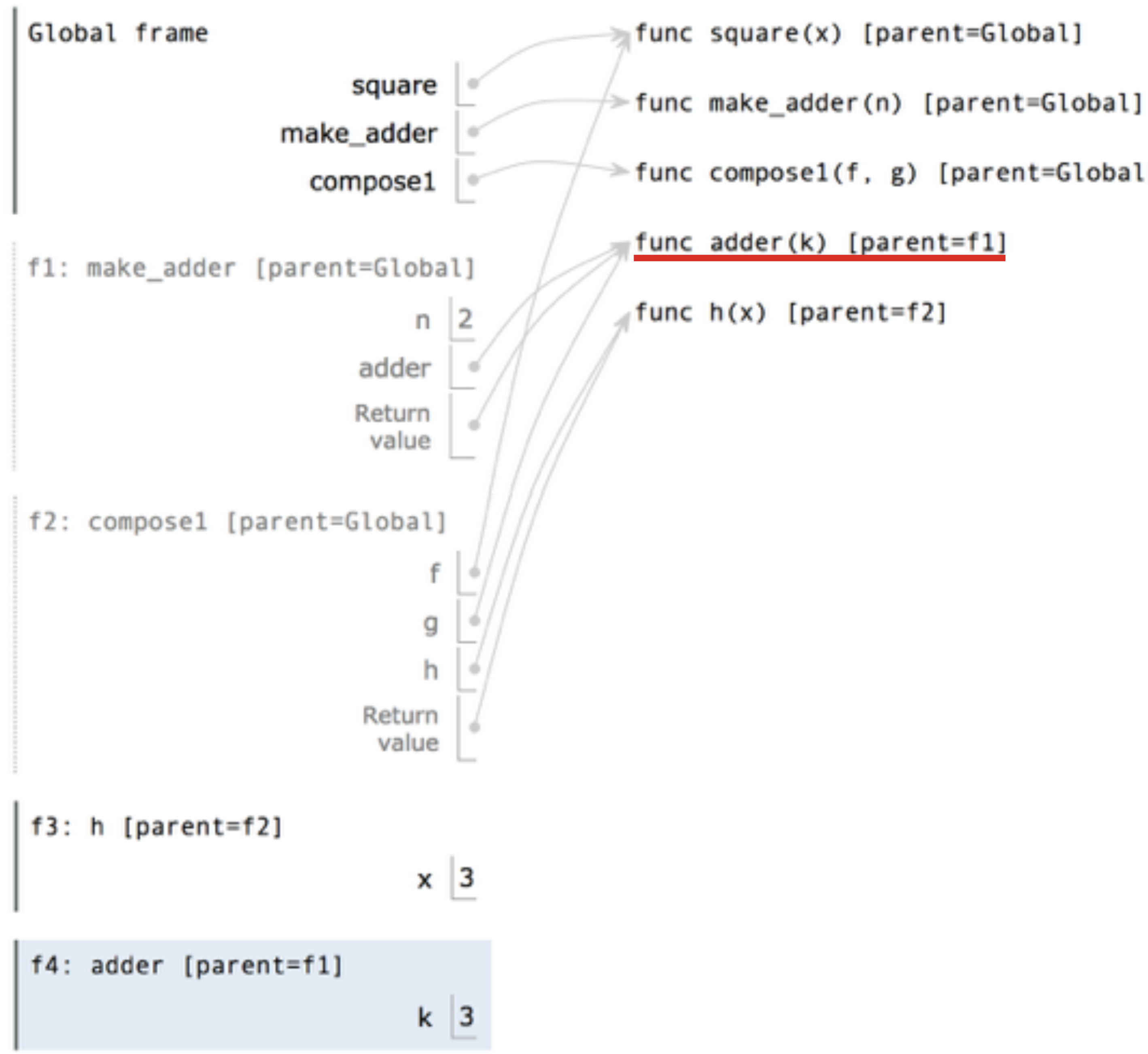

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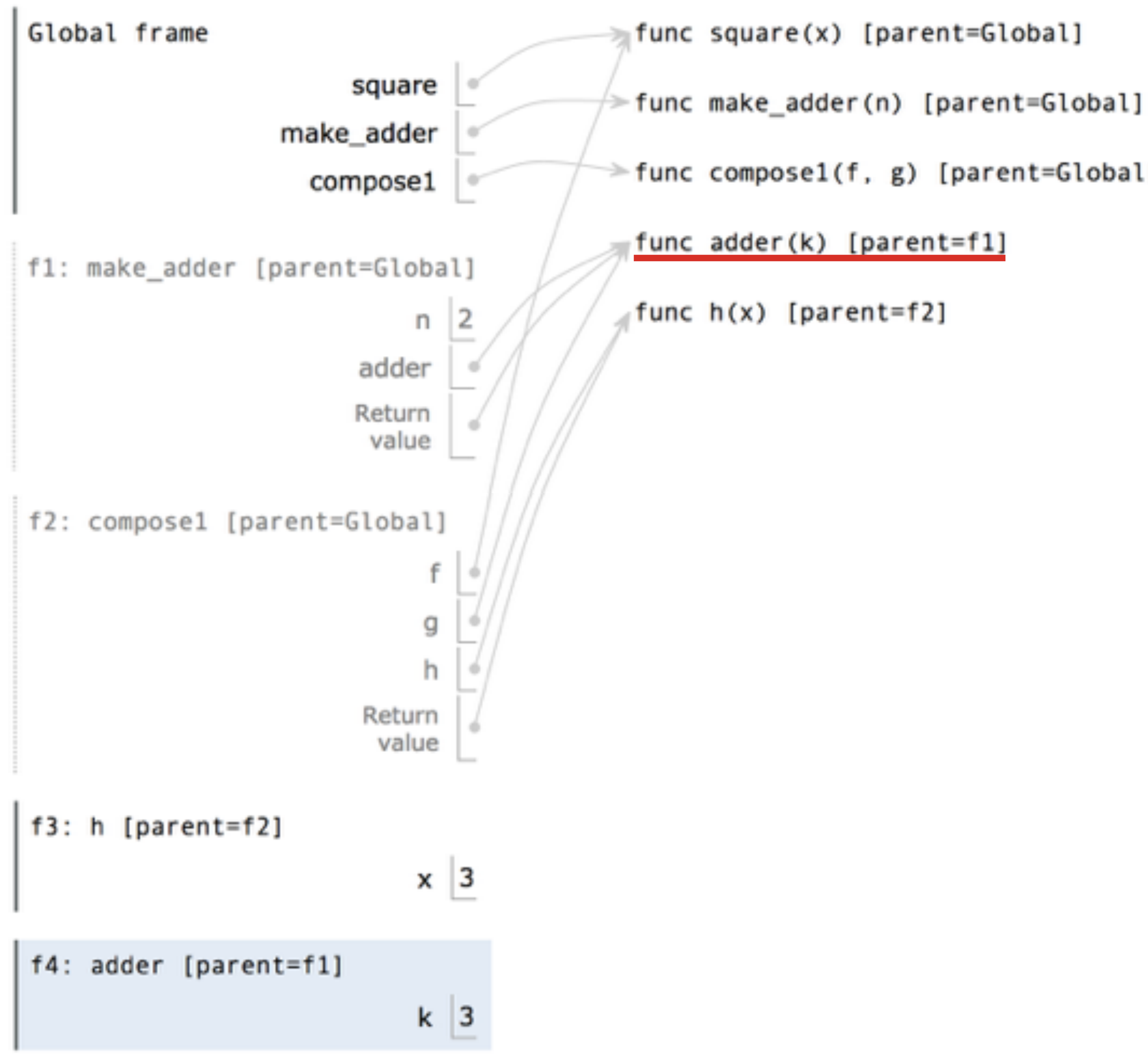

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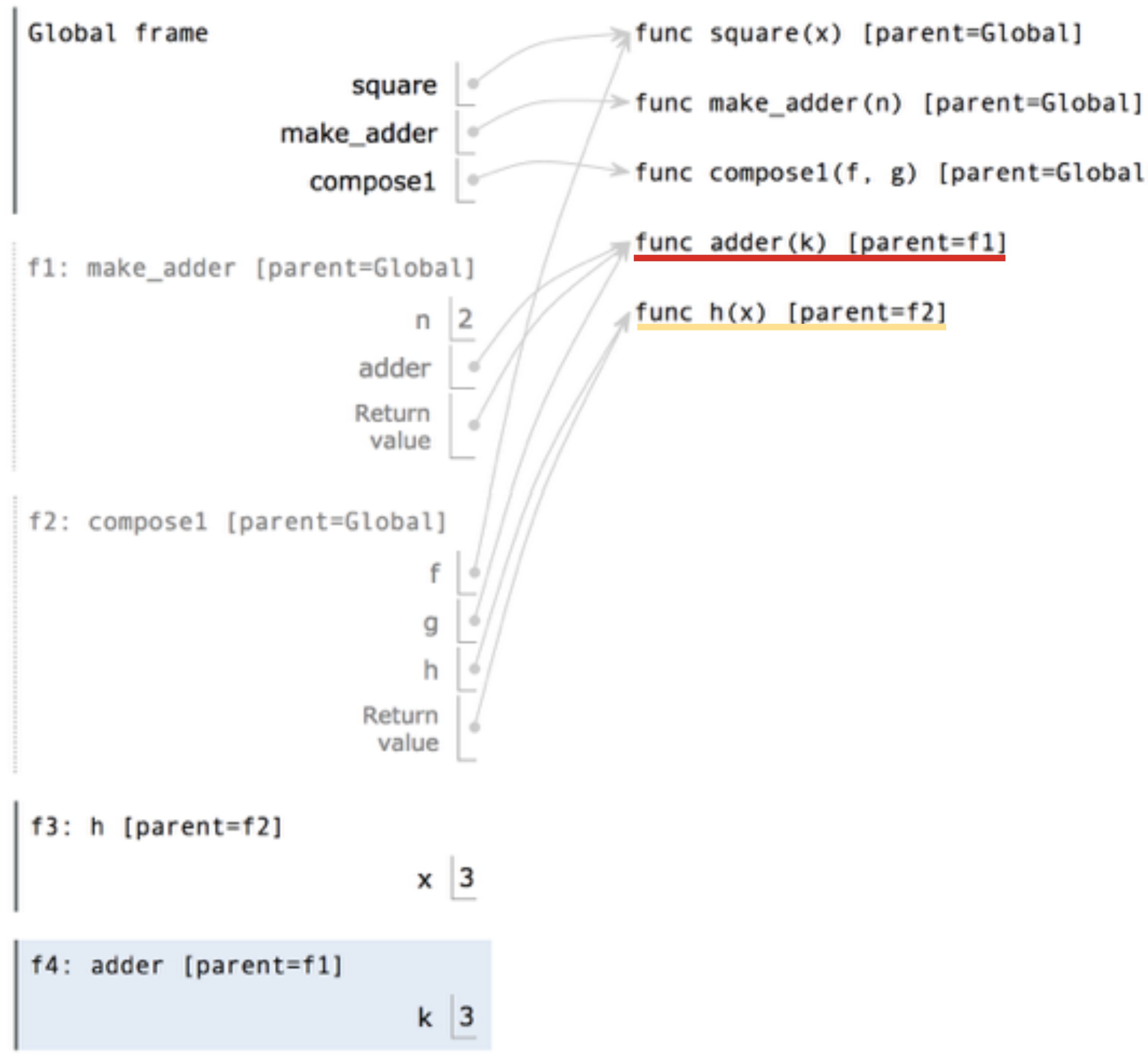

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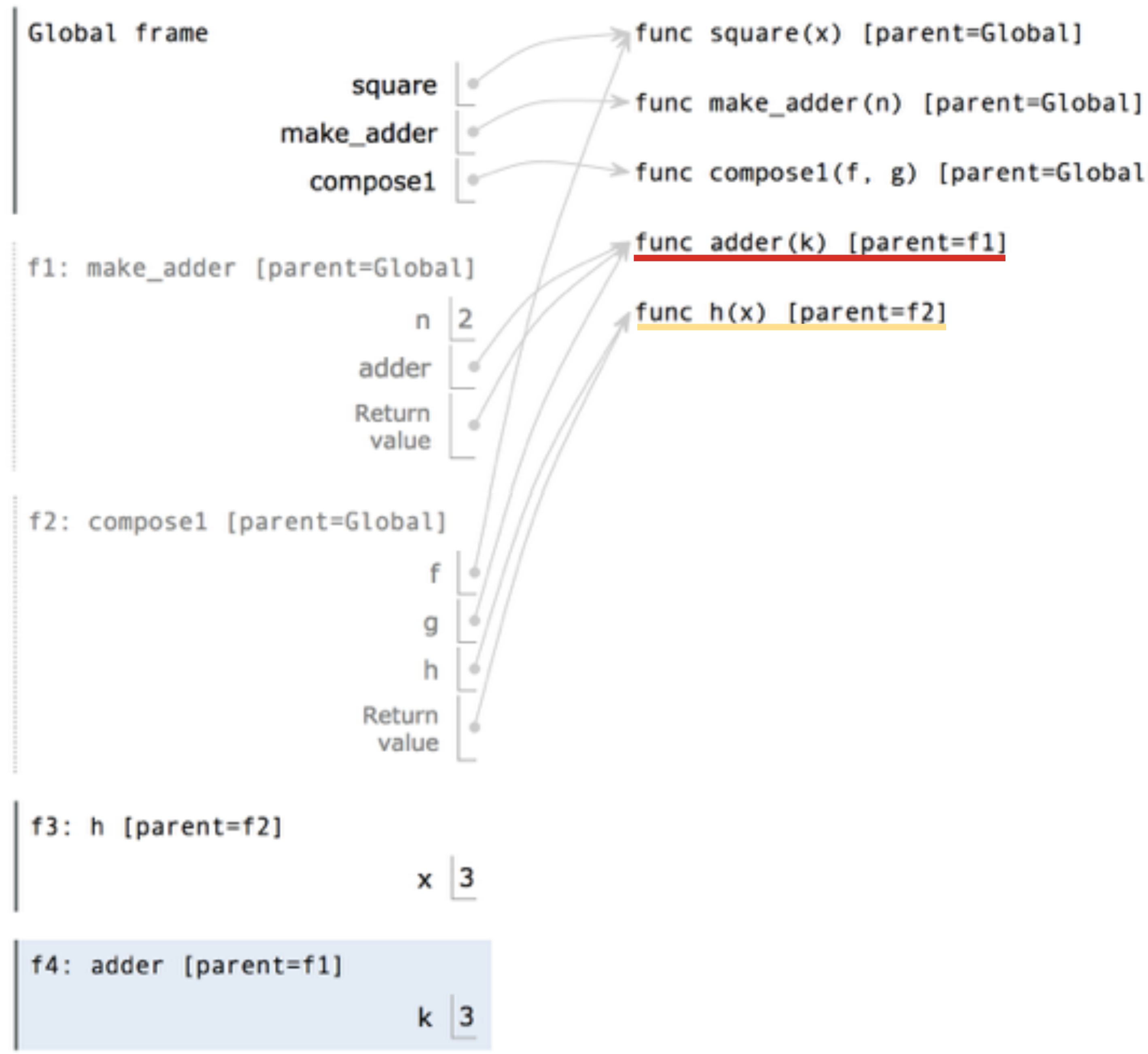

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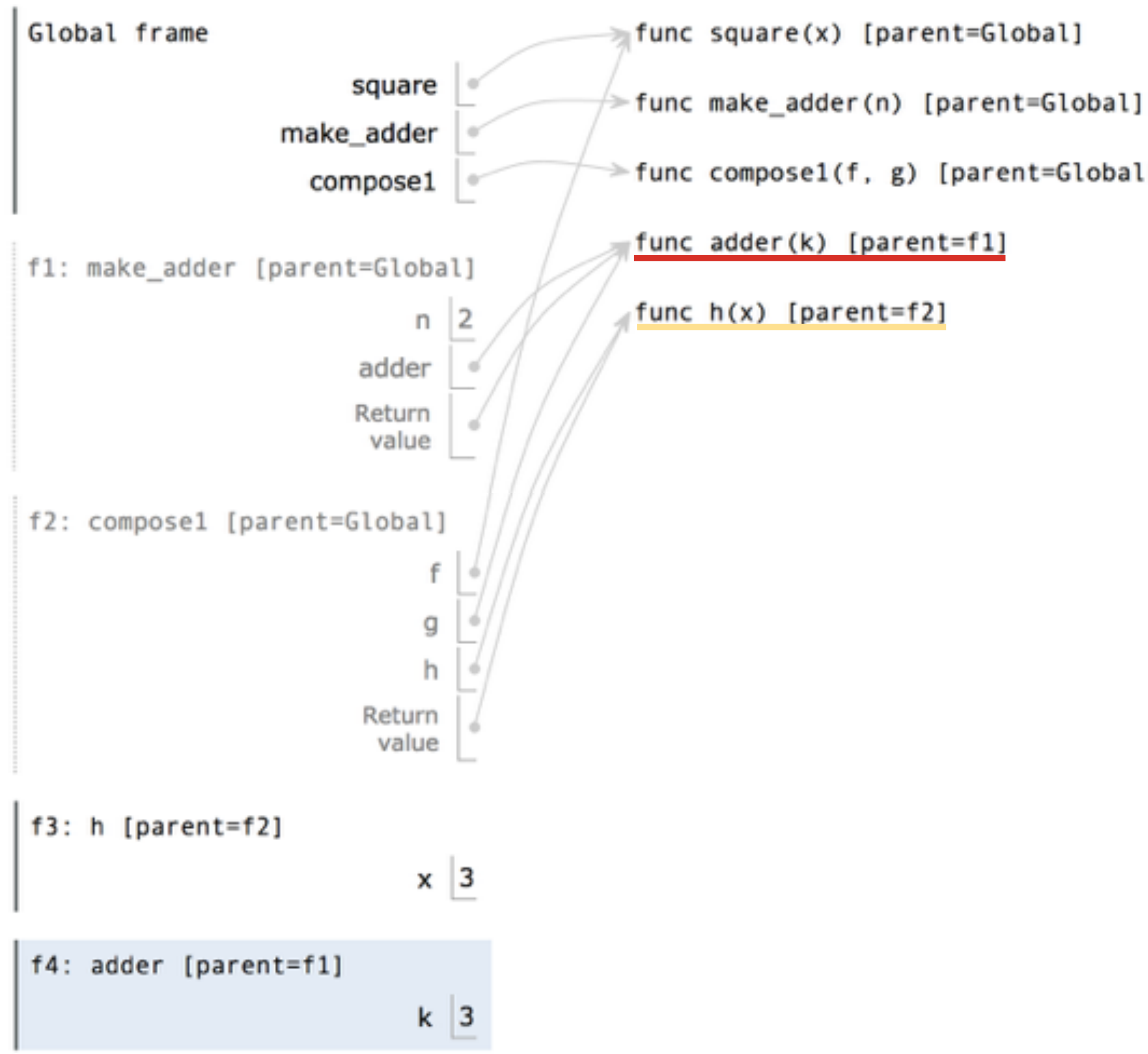

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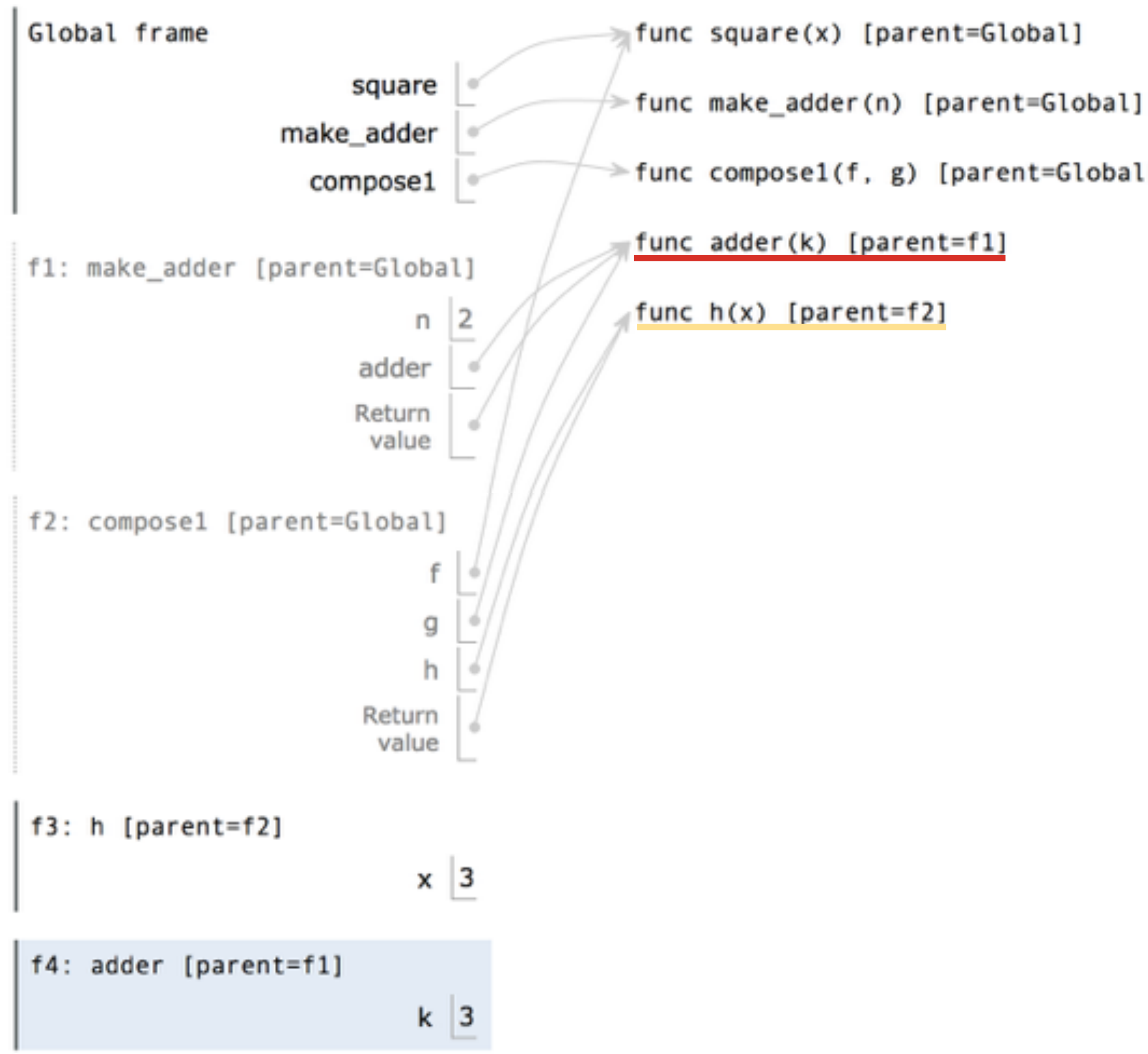

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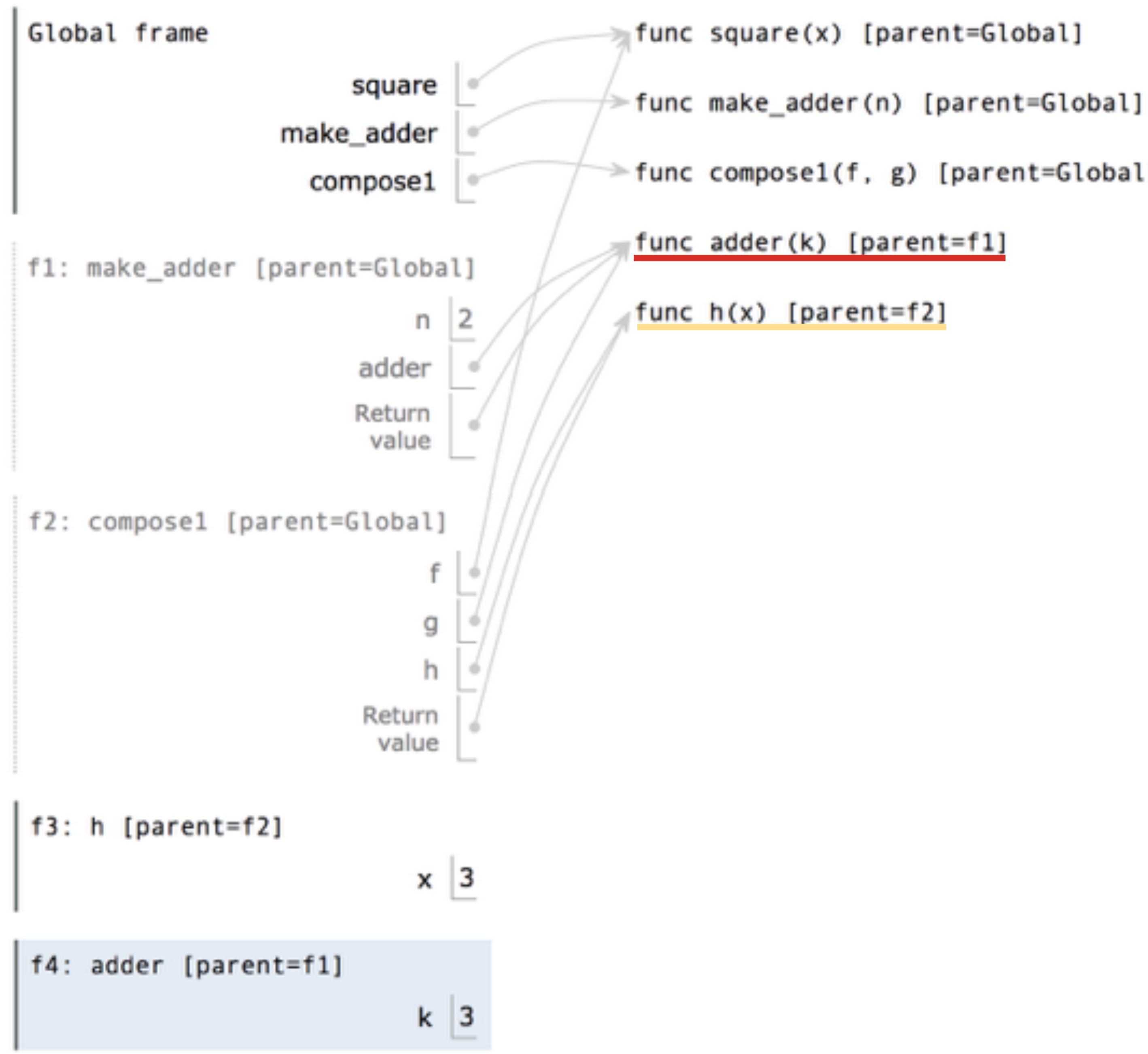

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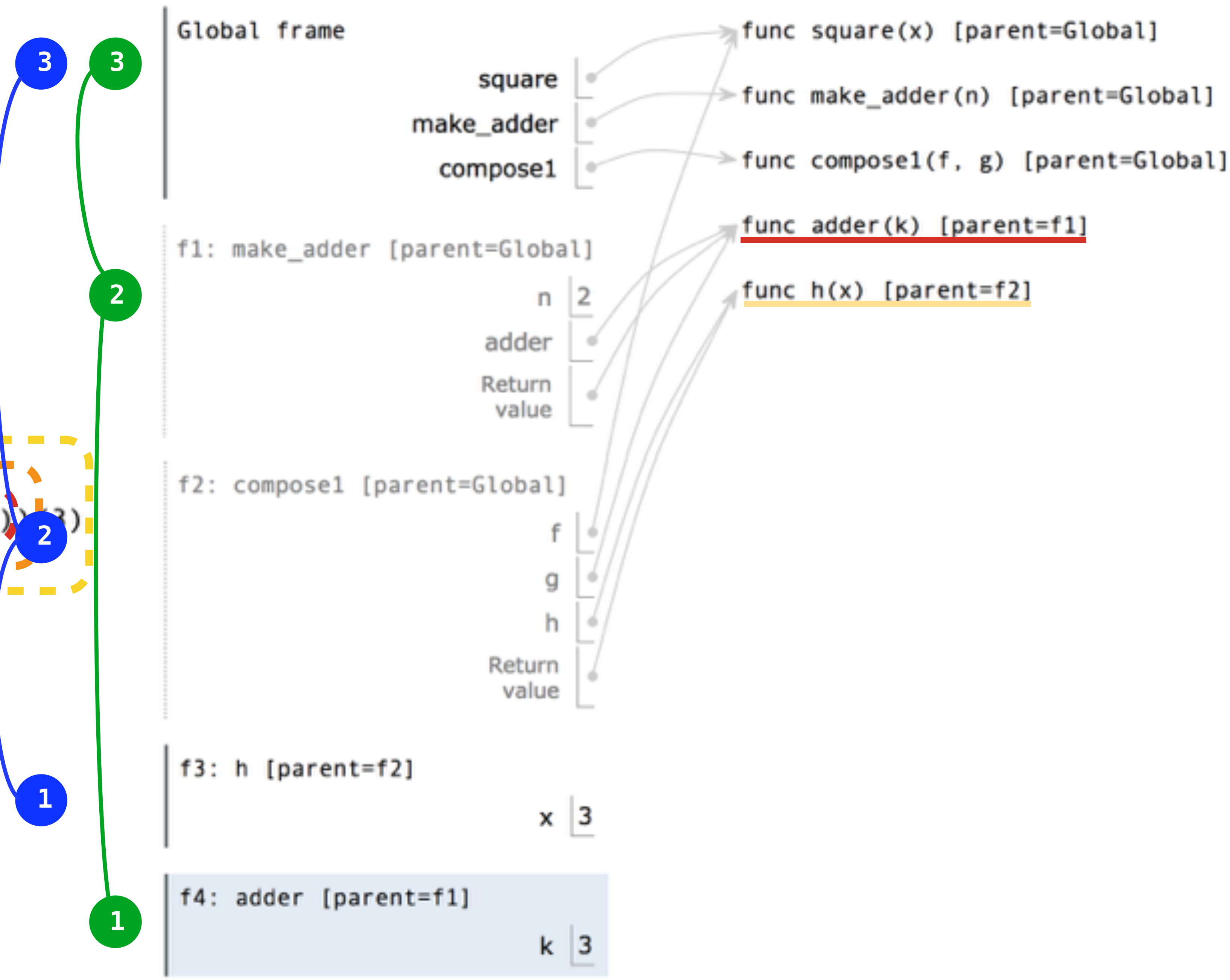

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- Currying allows us to represent functions with multiple variables as chains of functions with single variables
- It is named after mathematician and logician Haskell Curry (who rediscovered it after Moses Schönfinkel)

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```

Application: Currying

- add is a two-argument function that returns the sum of the two arguments
- make_adder is a one-argument function that returns a one-argument function that returns the sum of the two arguments
- Currying allows us to represent functions with multiple variables as chains of functions with single variables
- It is named after mathematician and logician Haskell Brooks Curry (who rediscovered it after Moses Schönfinkel)

```
(lambda x, y: x * y + 1)(3, 4)
(lambda x: lambda y: x * y + 1)(3)(4)
```