

1 Streams

In Python, we can use iterators to represent infinite sequences (for example, the generator for all natural numbers). However, Scheme does not support iterators. Let’s see what happens when we try to use a Scheme list to represent an infinite sequence of natural numbers:

```
scm> (define (naturals n)
    (cons n (naturals (+ n 1))))
naturals
scm> (naturals 0)
Error: maximum recursion depth exceeded
```

Because `cons` is a regular procedure and both its operands must be evaluated before the pair is constructed, we cannot create an infinite sequence of integers using a Scheme list. Instead, our Scheme interpreter supports streams, which are lazy Scheme lists. The first element is represented explicitly, but the rest of the stream’s elements are computed only when needed. Computing a value only when it’s needed is also known as lazy evaluation.

```
scm> (define (naturals n)
    (cons-stream n (naturals (+ n 1))))
naturals
scm> (define nat (naturals 0))
nat
scm> (car nat)
0
scm> (cdr nat)
#[promise (not forced)]
scm> (car (cdr-stream nat))
1
scm> (car (cdr-stream (cdr-stream nat)))
2
```

We use the special form `cons-stream` to create a stream:

```
(cons-stream <operand1> <operand2>)
```

`cons-stream` is a special form because the second operand is not evaluated when evaluating the expression. To evaluate this expression, Scheme does the following:

1. Evaluate the first operand.
2. Construct a promise containing the second operand.
3. Return a pair containing the value of the first operand and the promise.
To actually get the rest of the stream, we must call cdr-stream on it to force the promise to be evaluated. Note that this argument is only evaluated once and is then stored in the promise; subsequent calls to cdr-stream returns the value without recomputing it. This allows us to efficiently work with infinite streams like the naturals example above. We can see this in action by using a non-pure function to compute the rest of the stream:

```scheme
scm> (define (compute-rest n)
...>  (print 'evaluating!)
...>  (cons-stream n nil))
compute-rest
scm> (define s (cons-stream 0 (compute-rest 1)))
s
scm> (car (cdr-stream s))
evaluating!
1
scm> (car (cdr-stream s))
1
```

Here, the expression compute-rest 1 is only evaluated the first time cons-stream is called, so the symbol evaluating! is only printed the first time.

When displaying a stream, the first element of the stream and the promise are displayed separated by a dot (this indicates that they are part of the same pair, with the promise as the cdr). If the value in the promise has not been evaluated by calling cdr-stream, we consider it to be not forced. Otherwise, we consider it forced.

```scheme
scm> (define s (cons-stream 1 nil))
s
scm> s
(1 . #[promise (not forced)])
scm> (cdr-stream s) ; nil
()  
scm> s
(1 . #[promise (forced)])
```

Streams are very similar to Scheme lists in that they are also recursive structures. Just like the cdr of a Scheme list is either another Scheme list or nil, the cdr-stream of a stream is either a stream or nil. The difference is that whereas both arguments to cons are evaluated upon calling cons, the second argument to cons-stream isn’t evaluated until the first time that cdr-stream is called.

Here’s a summary of what we just went over:

- nil is the empty stream
- cons-stream constructs a stream containing the value of the first operand and a promise to evaluate the second operand
- car returns the first element of the stream
- cdr-stream computes and returns the rest of stream
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Questions

1.1 What would Scheme display?

As you work through these problems, remember that streams have two important components:

- Lazy evaluation – so the remainder of the stream isn’t computed until explicitly requested.
- Memoization – so anything we compute won’t be recomputed.

The examples here stretch these concepts to the limit. In most practical use cases, you may find you rarely need to redefine functions that compute the remainder of the stream.

```
(define (has-even? s)
  (cond ((null? s) #f)
        ((even? (car s)) #t)
        (else (has-even? (cdr-stream s)))))

(define (f x) (* 3 x))

(define nums (cons-stream 1 (cons-stream (f 3) (cons-stream (f 5) nil))))
```

```
scm> nums
(1 . #[promise (not forced)])
```

```
scm> (cdr nums)
#[promise (not forced)]
```

```
scm> (cdr-stream nums)
(9 . #[promise (not forced)])
```

```
scm> nums
(1 . #[promise (forced)])
```

```
scm> (define (f x) (* 2 x))
```

```
scm> (cdr-stream nums)
(9 . #[promise (not forced)])
```

```
scm> (cdr-stream (cdr-stream nums))
(10 . #[promise (not forced)])
```

```
scm> (has-even? nums)
True
```

Video walkthrough
Streams
1.2 Using streams can be tricky! Compare the following two implementations of `filter-stream`, the first is a correct implementation whereas the second is wrong in some way.

What's wrong with the second implementation?

; Correct
(define (filter-stream f s)
  (cond
   ((null? s) nil)
   ((f (car s)) (cons-stream (car s) (filter-stream f (cdr-stream s))))
   (else (filter-stream f (cdr-stream s))))
)

; Incorrect
(define (filter-stream f s)
  (if (null? s) nil
    (let ((rest (filter-stream f (cdr-stream s))))
      (if (f (car s))
        (cons-stream (car s) rest)
        rest)))))

Evaluating rest will result in infinite recursion if s is an infinite stream! In the body of filter-stream, rest is always computed before cons-stream can delay the evaluation.

Another way of thinking about this is that everything in the body of the let doesn't matter. All we will be doing is repeatedly doing the recursive call on filter-stream.

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1.3 Write a function `map-stream`, which takes a function f and a stream s. It returns a new stream which has all the elements from s, but with f applied to each one.

(define (map-stream f s)
  (if (null? s) nil
    (cons-stream (f (car s)) (map-stream f (cdr-stream s))))
)

It might help to also compare this to the version of map for regular (non-stream) Scheme lists:

(define (map f s)
  (if (null? s) nil
    (cons (f (car s)) (map f (cdr s)))))

Not too different, eh? The main change we've made is indicating we want to lazily evaluate the rest of our stream by using cons-stream instead of cons, and recognizing is a stream rather than a regular list by using cdr-stream.

s

1.4 Write a function `slice` which takes in a stream s, a start, and an end. It should return a Scheme list that contains the elements of s between index start and end,
not including end. If the stream ends before end, you can return nil.

\(\textbf{define} \ (\text{slice} \ s \ \text{start} \ \text{end})\)

\[
(\text{cond}
  
  ((\text{or} \ (\text{null?} \ s) \ (= \ \text{end} \ 0)) \ \text{nil})
  
  ((> \ \text{start} \ 0)
    \ (\text{slice} \ (\text{cdr-stream} \ s) \ (- \ \text{start} \ 1) \ (- \ \text{end} \ 1)))
  
  (\text{else}
    \ (\text{cons} \ (\text{car} \ s)
      \ (\text{slice} \ (\text{cdr-stream} \ s) \ (- \ \text{start} \ 1) \ (- \ \text{end} \ 1)))))
\]

scm\> \ (\textbf{define} \ nat \ (\text{naturals} \ 0)) \ ; \text{See naturals procedure defined earlier}

nat

scm\> \ (\text{slice} \ nat \ 4 \ 12)

(4 5 6 7 8 9 10 11)
We can even represent the sequence of all prime numbers as an infinite stream! Define a function `sieve`, which takes in a stream of increasing numbers and returns a stream containing only those numbers which are not multiples of an earlier number in the stream. We can define `primes` by sifting all natural numbers starting at 2. Look online for the Sieve of Eratosthenes if you need some inspiration.

```
(define (sieve s)
  (cons-stream
    (car s)
    (sieve (sift (car s) (cdr-stream s))))
)

(define (sift prime s)
  (filter-stream
    (lambda (x) (not (= 0 (modulo x prime))))
    s)
)

(define primes
  (sieve (naturals 2)))

scm> (slice primes 0 10)
(2 3 5 7 11 13 17 19 23 29)
```

Since streams only evaluate the next element when they are needed, we can combine infinite streams together for interesting results! Use it to define a few of our favorite sequences. We've defined the function `combine-with` for you below, as well as an example of how to use it to define the stream of even numbers.

```
(define (combine-with f xs ys)
  (if (or (null? xs) (null? ys))
      nil
      (cons-stream
        (f (car xs) (car ys))
        (combine-with f (cdr-stream xs) (cdr-stream ys))))
)

scm> (define evens (combine-with + (naturals 0) (naturals 0)))
evans

scm> (slice evens 0 10)
(0 2 4 6 8 10 12 14 16 18)
```

For these questions, you may use the `naturals` stream in addition to `combine-with`.

i. (define factorials

```
(cons-stream 1 (combine-with * (naturals 1) factorials)))
```

```
scm> (slice factorials 0 10)
(1 1 2 6 24 120 720 5040 40320 362880)
```

ii. (define fibs

```
(cons-stream 0
  (cons-stream 1
    (combine-with + fibs (cdr-stream fibs))))
```

```
scm> (slice fibs 0 10)
(0 1 1 2 3 5 8 13 21 34)
```
iii. (Extra for practice) Write $\text{exp}$, which returns a stream where the $n$th term represents the degree-$n$ polynomial expansion for $e^x$, which is $\sum_{i=0}^{n} \frac{x^i}{i!}$.

You may use $\text{factorials}$ in addition to $\text{combine-with}$ and $\text{naturals}$ in your solution.

(define (exp x)

    (let ((terms (combine-with (lambda (a b) (/ (expt x a) b))
                                  (cdr-stream (naturals 0)))
           (cdr-stream (factorials))))

       (cons-stream 1 (combine-with + terms (exp x)))))

scm> (slice (exp 2) 0 5)
(1 3 5 6.333333333 7)
Extra Questions

1.1 In Discussion 9, we saw how to write a macro that creates a lambda function given an expression. While creating a parameter-less function might not seem that useful at first, it can be helpful in many cases when we don’t want to immediately evaluate an expression.

Using the make-lambda macro you defined in Discussion 9, define make-stream, a macro which returns a pair of elements, where the second element is not evaluated until cdr-stream is called on it. Also define the procedure cdr-stream, which takes in a stream returned by make-stream and returns the result of evaluating the second element in the stream pair.

Unlike the streams we’ve seen in lecture and earlier in discussion, if you repeatedly call cdr-stream on a stream returned by make-stream, you may evaluate an expression multiple times.

(define-macro (make-stream first second))

(define (cdr-stream stream))

(scm> (define a (make-stream (print 1) (make-stream (print 2) nil))))

1
a

(scm> (define b (cdr-stream a)))

2
b

(scm> (cdr-stream b))

()