Box-and-Pointer Notation

The Closure Property of Data Types

- A method for combining data values satisfies the *closure property* if:

  The result of combination can itself be combined using the same method

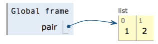
- Closure is powerful because it permits us to create hierarchical structures
- Hierarchical structures are made up of parts, which themselves are made up of parts, and so on

Lists can contain lists as elements (in addition to anything else)

Box-and-Pointer Notation in Environment Diagrams

Lists are represented as a row of index-labeled adjacent boxes, one per element

Each box either contains a primitive value or points to a compound value




```
pair = [1, 2]
```

Interactive Diagram

Box-and-Pointer Notation in Environment Diagrams

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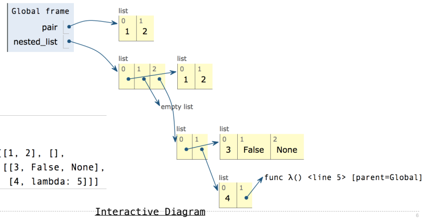



```
pair = [1, 2]

nested_list = [[1, 2], [],
               [[3, False, None],
                [4, lambda: 5]]]
```

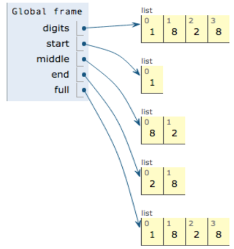
Interactive Diagram

Slicing

(Demo)

Slicing Creates New Values

```
digits = [1, 8, 2, 8]
start = digits[:1]
middle = digits[1:3]
end = digits[2:]
full = digits[:]
```



Interactive Diagram

Processing Container Values

Sequence Aggregation

Several built-in functions take iterable arguments and aggregate them into a value

- **sum**(iterable[, start]) -> value

  Return the sum of an iterable of numbers (NOT strings) plus the value of parameter 'start' (which defaults to 0). When the iterable is empty, return start.

- **max**(iterable[, key=func]) -> value
  max(a, b, c, ...[, key=func]) -> value

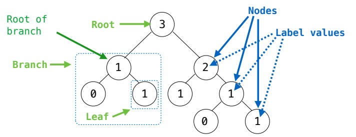
  With a single iterable argument, return its largest item.
  With two or more arguments, return the largest argument.

- **all**(iterable) -> bool

  Return True if bool(x) is True for all values x in the iterable.
  If the iterable is empty, return True.

Trees

Tree Abstraction



Recursive description (wooden trees):

A **tree** has a **root** and a list of **branches**

Each branch is a **tree**

A tree with zero branches is called a **leaf**

Relative description (family trees):

Each location in a tree is called a **node**

Each **node** has a **label value**

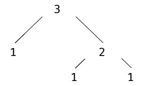
One node can be the **parent/child** of another

People often refer to values by their locations: "each parent is the sum of its children"

Implementing the Tree Abstraction

```
def tree(label, branches=[]):
    return [label] + branches

def label(tree):
    return tree[0]

def branches(tree):
    return tree[1:]
```

- A tree has a label value and a list of branches



```
>>> tree(3, [tree(1),
...          tree(2, [tree(1),
...                   tree(1)])])
[3, [1], [2, [1], [1]]]
```

Implementing the Tree Abstraction

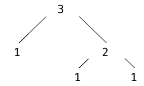
```
def tree(label, branches=[]):
    for branch in branches:
        assert is_tree(branch)
    return [label] + list(branches)

def label(tree):
    return tree[0]

def branches(tree):
    return tree[1:]

def is_tree(tree):
    if type(tree) != list or len(tree) < 1:
        return False
    for branch in branches(tree):
        if not is_tree(branch):
            return False
    return True
```

Verifies the tree definition

Creates a list from a sequence of branches

Verifies that tree is bound to a list

- A tree has a label value and a list of branches



```
>>> tree(3, [tree(1),
...          tree(2, [tree(1),
...                   tree(1)])])
[3, [1], [2, [1], [1]]]

def is_leaf(tree):
    return not branches(tree)
```

(Demo)

Tree Processing

(Demo)

Tree Processing Uses Recursion

Processing a leaf is often the base case of a tree processing function

The recursive case typically makes a recursive call on each branch, then aggregates

```
def count_leaves(t):
    """Count the leaves of a tree."""
    if is_leaf(t):
        return 1
    else:
        branch_counts = [count_leaves(b) for b in branches(t)]
        return sum(branch_counts)
```

(Demo)

Discussion Question

Implement leaves, which returns a list of the leaf labels of a tree

Hint: If you sum a list of lists, you get a list containing the elements of those lists

```
>>> sum([ [1], [2, 3], [4] ], [])
[1, 2, 3, 4]
>>> sum([ [1] ], [])
[1]
>>> sum([ [[1]], [2] ], [])
[[1], 2]
```

```
def leaves(tree):
    """Return a list containing the leaves of tree.

    >>> leaves(fib_tree(5))
    [1, 0, 1, 0, 1, 1, 0, 1]
    """
    if is_leaf(tree):
        return [label(tree)]
    else:
        return sum(List of leaves for each branch, []))
```

branches(tree)	[b for b in branches(tree)]
leaves(tree)	[s for s in leaves(tree)]
[branches(b) for b in branches(tree)]	[branches(s) for s in leaves(tree)]
[leaves(b) for b in branches(tree)]	[leaves(s) for s in leaves(tree)]

Creating Trees

A function that creates a tree from another tree is typically also recursive

```
def increment_leaves(t):
    """Return a tree like t but with leaf values incremented."""
    if is_leaf(t):
        return tree(label(t) + 1)
    else:
        bs = [increment_leaves(b) for b in branches(t)]
        return tree(label(t), bs)

def increment(t):
    """Return a tree like t but with all node values incremented."""
    return tree(label(t) + 1, [increment(b) for b in branches(t)])
```

Example: Printing Trees

(Demo)