What Are the Questions?

- ncipal concern throughout engineering:
- eer is someone who can do for a dime what any fool
 a dollar."
- an
- al cost (for programs, time to run, space requirements).
- ent costs: How much engineering time? When deliv-
- failure: How robust? How safe?
- ram fast enough? Depends on:
 - *purpose;*
 - *ut data.*
- pace (memory, disk space)?
- ends on what input data.
- cale, as input gets big?

Cost Measures (Time)

- r execution time
- o this at home:
 `e java FindPrimes 1000`
- es: easy to measure, meaning is obvious.
- te where time is critical (real-time systems, e.g.).
- ages: applies only to specific data set, compiler, ma-
 .
- imes certain statements are executed:
- es: more general (not sensitive to speed of machine).
- ages: doesn't tell you actual time, still applies only to
 ata sets.
- ecution times:
- *ormulas* for execution times as functions of input size.
- es: applies to all inputs, makes scaling clear.
- age: practical formula must be approximate, may tell
 about actual time.

Handy Tool: Order Notation

- try to produce *specific* functions that specify size, but
 ies of similar functions.
- ng like "f is bounded by g if it is in g's family."
- tion $g(x)$, the functions $2g(x)$, $1000g(x)$, or for any $K >$
 ll have the same "shape". So put all of them into g's
- $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some
 has g's shape "except for small values." So put all of
 family.
- pper limits, throw in all functions that are everywhere
 ber of g's family. Call this family $O(g)$ or $O(g(n))$.
- nt lower limits, throw in all functions that are every-
 e member of g's family. Call this family $\Omega(g)$.
- e $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed*
 of g's family.

CS61B Lecture #15: Complexity

Enlightening Example

- a text corpus (say 10^7 bytes or so), and find and print
 quently used words, together with counts of how often
- nuth): Heavy-Duty data structures
 - implementation, randomized placement, pointers ga-
 ral pages long.
- oug McIlroy): UNIX shell script:

```
 '[:alpha:]' '[\n*]' < FILE | \
 \
r -k 1,1 | \
```

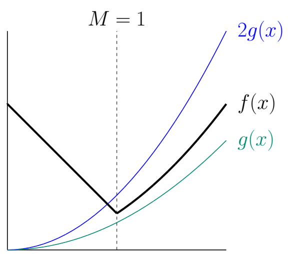
- ter?
 - h faster,
 - ok 5 minutes to write and processes 20MB in 1 minute.
 .
- s, anything will do: Keep It Simple.

Asymptotic Cost

- ecution time lets us see *shape* of the cost function.
- e approximating anyway, pointless to be precise about
 s:
- on small inputs:
 - ays pre-calculate some results.
 - or small inputs not usually important.
- factors (as in "off by factor of 2"):
 - anging machines causes constant-factor change.
- ract away from (i.e., ignore) these things?

Big Oh

y bounding from above.



$2g(x)$ as long as $x > 1$,

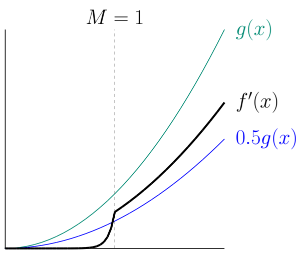
g's upper-bound family, written

$$f(x) \in O(g(x)),$$

gh $f(x) > g(x)$ everywhere.

Big Omega

y bounding from below:



$\frac{1}{2}g(x)$ as long as $x > 1$,

g's lower-bound family, written

$$f'(x) \in \Omega(g(x)),$$

gh $f(x) < g(x)$ everywhere.

also have $f'(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$, so

$$f(x), f'(x) \in \Theta(g(x)).$$

Why It Matters

ientists often talk as if constant factors didn't matter he difference of $\Theta(N)$ vs. $\Theta(N^2)$.

ey do, but at some point, constants always get swamped.

$\sqrt{n}$	n	$n \lg n$	n^2	n^3	2^n
1.4	2	2	4	8	4
2	4	8	16	64	16
2.8	8	24	64	512	256
4	16	64	256	4,096	65,636
5.7	32	160	1024	32,768	4.2×10^9
8	64	384	4,096	262,144	1.8×10^{19}
11	128	896	16,384	2.1×10^9	3.4×10^{38}
⋮	⋮	⋮	⋮	⋮	⋮
32	1,024	10,240	1.0×10^6	1.1×10^9	1.8×10^{308}
⋮	⋮	⋮	⋮	⋮	⋮
1024	1.0×10^6	2.1×10^7	1.1×10^{12}	1.2×10^{18}	$6.7 \times 10^{315,652}$

ne Intuition on Meaning of Growth

oblem can you solve in a given time?

wing table, left column shows time in microseconds to problem as a function of problem size N.

the *size of problem* that can be solved in a second, (31 days), and century, for various relationships be-equired and problem size.

size

c) for ze N	Max N Possible in 1 second	1 hour	1 month	1 century
	10^{300000}	$10^{1000000000}$	$10^{8 \cdot 10^{11}}$	$10^{9 \cdot 10^{14}}$
	10^6	$3.6 \cdot 10^9$	$2.7 \cdot 10^{12}$	$3.2 \cdot 10^{15}$
	63000	$1.3 \cdot 10^8$	$7.4 \cdot 10^{10}$	$6.9 \cdot 10^{13}$
	1000	60000	$1.6 \cdot 10^6$	$5.6 \cdot 10^7$
	100	1500	14000	150000
	20	32	41	51

Using the Notation

order notation for any kind of real-valued function.

hem to describe cost functions. Example:

```
position of X in list L, or -1 if not found. */
(List L, Object X) {

 = 0; L != null; L = L.next, c += 1)
 (X.equals(L.head)) return c;
 -1;
```

esentative operation: number of .equals tests.

th of L, then loop does *at most* N tests: *worst-case* sts.

al # of instructions executed is roughly proportional worst case, so can also say worst-case time is $O(N)$, f units used to measure.

provision (in defn. of $O(\cdot)$) to handle empty list.

Be Careful

that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ bounds are loose.

ase time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* e loop *always* takes time N, or even $K \cdot N$ for some K.

are just saying something about the *function* that maps *largest possible* time required to process an array of

ch as possible about our worst-case time, we should try ound: in this case, we can: $\Theta(N)$.

at still tells us nothing about *best-case* time, which n we find X at the beginning of the loop. Best-case time

rsion and Recurrences: Fast Growth

e of recursion. In the worst case, both recursive calls

```
ff X is a substring of S */
curs(String S, String X) {
uals(X)) return true;
ngth() <= X.length()) return false;

(S.substring(1), X) ||
(S.substring(0, S.length()-1), X);
```

) to be the worst-case cost of occurs(S,X) for S of
f fixed size N_0, measured in # of calls to occurs. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

ws exponentially:

$$N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0}\cdot 1+1)+\ldots+1$$

$$N_0 + 2^{N-N_0-1} + 2^{N-N_0-2} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$

other Typical Pattern: Merge Sort

```
: L) {
n() < 2) return L;
) and L1 of about equal size;
));   L1 = sort(L1);
e of L0 and L1
```

Merge ("combine into a single ordered list") takes time proportional to size of its result.

at size of L is $N = 2^k$, worst-case cost function, $C(N)$,
r merge time ($\propto$ # items merged):

$$\begin{aligned} C(N) &= \begin{cases} 1, & \text{if } N < 2; \\ 2C(N/2)+N, & \text{if } N \ge 2. \end{cases} \\ &= 2(2C(N/4)+N/2)+N \\ &= 4C(N/4)+N+N \\ &= 8C(N/8)+N+N+N \\ &= N\cdot 1 + \underbrace{N+N+\ldots+N}_{k=\lg N} \\ &= N + N\lg N \in \Theta(N\lg N) \end{aligned}$$

$)(N\lg N)$ for arbitrary N (not just 2^k).

Effect of Nested Loops

s often lead to polynomial bounds:

```
 i = 0; i < A.length; i += 1)
nt j = 0; j < A.length; j += 1)
(i != j && A[i] == A[j])
return true;
lse;
```

: is $O(N^2)$, where N = A.length. *Worst-case time* is

icient though:

```
 i = 0; i < A.length; i += 1)
nt j = i+1; j < A.length; j += 1)
(A[i] == A[j]) return true;
lse;
```

ase time is proportional to

$$-1+N-2+\ldots+1 = N(N-1)/2 \in \Theta(N^2)$$

ic time unchanged by the constant factor).

Binary Search: Slow Growth

```
 is an element of S[L .. U]. Assumes
ding order, 0 <= L <= U-1 < S.length. */
tring X, String[] S, int L, int U) {
eturn false;
)/2;
 X.compareTo(S[M]);
 0) return isIn(X, S, L, M-1);
ect > 0) return isIn(X, S, M+1, U);
true;
```

-case time, $C(D)$, (as measured by # of string compar-
ds on size $D = U - L + 1$.

e S[M] from consideration each time and look at half the
e $D = 2^k - 1$ for simplicity, so:

$$\begin{aligned} C(D) &= \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases} \\ &= \underbrace{1+1+\ldots+1}_{k}+0 \\ &= k = \lceil \lg D \rceil \in \Theta(\lg D) \end{aligned}$$