## CS61B Lecture #16: Complexity

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contest 14 October. Details to follow.

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## What Are the Questions?

ncipal concern throughout engineering:

eer is someone who can do for a dime what any fool
a dollar."

n

al cost (for programs, time to run, space requirements).

ent costs: How much engineering time? When deliv-

ailure: How robust? How safe?

am fast enough? Depends on:

*purpose;*

*t data.*

ace (memory, disk space)?

ends on what input data.

ale, as input gets big?

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## Enlightening Example

a text corpus (say $10^7$ bytes or so), and find and print
quently used words, together with counts of how often

nuth): Heavy-Duty data structures

implementation, randomized placement, pointers ga-
ral pages long.

oug McIlroy): UNIX shell script:

```
 '[:alpha:]' '[\n*]' < FILE | \

 \
r -k 1,1 | \
```

ter?

h faster,

ok 5 minutes to write and processes 30MB in $<$ 8 sec.

s, anything will do: Keep It Simple.

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## Cost Measures (Time)

execution time

this at home:

`java FindPrimes 1000`

s: easy to measure, meaning is obvious.

te where time is critical (real-time systems, e.g.).

ages: applies only to specific data set, compiler, ma-

imes certain statements are executed:

s: more general (not sensitive to speed of machine).

ages: doesn't tell you actual time, still applies only to
ata sets.

cution times:

ormulas for execution times as functions of input size.

s: applies to all inputs, makes scaling clear.

age: practical formula must be approximate, may tell
about actual time.

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## Asymptotic Cost

cution time lets us see *shape* of the cost function.

approximating anyway, pointless to be precise about
s:

n small inputs:

ays pre-calculate some results.

or small inputs not usually important.

factors (as in "off by factor of 2"):

anging machines causes constant-factor change.

act away from (i.e., ignore) these things?

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## Handy Tool: Order Notation

ry to produce *specific* functions that specify size, but
*es of similar* functions.

g like "$f$ is bounded by $g$ if it is in $g$'s family."

tion $g(x)$, the functions $2g(x)$, $1000g(x)$, or for any $K >$
ll have the same "shape". So put all of them into $g$'s

$h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some
has $g$'s shape "except for small values." So put all of
amily.

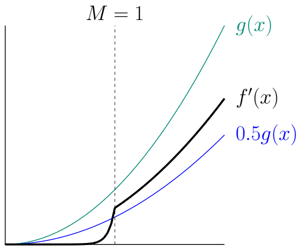
pper limits, throw in all functions that are everywhere
ber of $g$'s family. Call this family $O(g)$ or $O(g(n))$.

nt lower limits, throw in all functions that are every-
e member of $g$'s family. Call this family $\Omega(g)$.

e $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed*
ers of $g$'s family.

## Big Omega

y bounding from below:



$\frac{1}{2}g(x)$ as long as $x > 1$,

$g$'s lower-bound family, written

$$f'(x) \in \Omega(g(x)),$$

gh $f(x) < g(x)$ everywhere.

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## How We Use Order Notation

mathematics, you'll see $O(\ldots)$, etc., used generally to
ds on functions.

$$\pi(N) = \Theta(\frac{N}{\ln N})$$

d prefer to write

$$\pi(N) \in \Theta(\frac{N}{\ln N})$$

is the number of primes less than or equal to $N$.)

ze things like

$$f(x) = x^3 + x^2 + O(x),$$

$f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

oses, the functions we will be bounding will be *cost func-*
ons that measure the amount of execution time or the
ace required by a program or algorithm.

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## ne Intuition on Meaning of Growth

oblem can you solve in a given time?

wing table, left column shows time in microseconds to
problem as a function of problem size $N$.

the *size of problem* that can be solved in a second,
(31 days), and century, for various relationships be-
equired and problem size.

size

| c) for ze $N$ | Max $N$ Possible in 1 second | 1 hour | 1 month | 1 century |
|---|---|---|---|---|
| | $10^{300000}$ | $10^{1000000000}$ | $10^{8\cdot 10^{11}}$ | $10^{9\cdot 10^{14}}$ |
| | $10^6$ | $3.6 \cdot 10^9$ | $2.7 \cdot 10^{12}$ | $3.2 \cdot 10^{15}$ |
| / | $63000$ | $1.3 \cdot 10^8$ | $7.4 \cdot 10^{10}$ | $6.9 \cdot 10^{13}$ |
| | $1000$ | $60000$ | $1.6 \cdot 10^6$ | $5.6 \cdot 10^7$ |
| | $100$ | $1500$ | $14000$ | $150000$ |
| | $20$ | $32$ | $41$ | $51$ |

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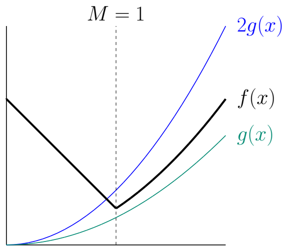
## Big Oh

y bounding from above.



$2g(x)$ as long as $x > 1$,

$g$'s upper-bound family, written

$$f(x) \in O(g(x)),$$

gh (in this case) $f(x) > g(x)$ everywhere.

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## Big Theta

previous slides, we not only have $f(x) \in O(g(x))$ and
$))$,...

$(x) \in \Omega(g(x))$ and $f'(x) \in O(g(x))$.

harize this all by saying $f(x) \in \Theta(g(x))$ and $f'(x) in \Theta(g(x))$.

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## Why It Matters

ientists often talk as if constant factors didn't matter
he difference of $\Theta(N)$ vs. $\Theta(N^2)$.

ey do matter, but at some point, constants always get

| $\sqrt{n}$ | $n$ | $n \lg n$ | $n^2$ | $n^3$ | $2^n$ |
|---|---|---|---|---|---|
| $1.4$ | $2$ | $2$ | $4$ | $8$ | $4$ |
| $2$ | $4$ | $8$ | $16$ | $64$ | $16$ |
| $2.8$ | $8$ | $24$ | $64$ | $512$ | $256$ |
| $4$ | $16$ | $64$ | $256$ | $4,096$ | $65,636$ |
| $5.7$ | $32$ | $160$ | $1024$ | $32,768$ | $4.2 \times 10^9$ |
| $8$ | $64$ | $384$ | $4,096$ | $262,144$ | $1.8 \times 10^{19}$ |
| $11$ | $128$ | $896$ | $16,384$ | $2.1 \times 10^9$ | $3.4 \times 10^{38}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $32$ | $1,024$ | $10,240$ | $1.0 \times 10^6$ | $1.1 \times 10^9$ | $1.8 \times 10^{308}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $1024$ | $1.0 \times 10^6$ | $2.1 \times 10^7$ | $1.1 \times 10^{12}$ | $1.2 \times 10^{18}$ | $6.7 \times 10^{315,652}$ |

## Be Careful

e that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ bounds are loose.

ase time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* e loop *always* takes time $N$, or even $K \cdot N$ for some $K$.

are just saying something about the *function* that maps *largest possible* time required to process an array of

ch as possible about our worst-case time, we should try ound: in this case, we can: $\Theta(N)$.

hat still tells us nothing about *best-case* time, which n we find `X` at the beginning of the loop. Best-case time

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## rsion and Recurrences: Fast Growth

e of recursion. In the worst case, both recursive calls

```
ff X is a substring of S */
curs(String S, String X) {
uals(X)) return true;
ngth() <= X.length()) return false;

(S.substring(1), X) ||
(S.substring(0, S.length()-1), X);
```

) to be the worst-case cost of `occurs(S,X)` for `S` of f fixed size $N_0$, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

ws exponentially:

$$N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0}\cdot 1+1)+\ldots+1$$

$$N_0} + 2^{N-N_0-1} + 2^{N-N_0-2} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$

---

## ther Typical Pattern: Merge Sort

```
 L) {
() < 2) return L;
 and L1 of about equal size;
);  L1 = sort(L1);
e of L0 and L1
```

Merge ("combine into a single ordered list") takes time proportional to size of its result.

at size of `L` is $N = 2^k$, worst-case cost function, $C(N)$, merge time ($\propto$ # items merged):

$$\begin{aligned} C(N) &= \begin{cases} 1, & \text{if } N < 2; \\ 2C(N/2)+N, & \text{if } N \ge 2. \end{cases} \\ &= 2(2C(N/4)+N/2)+N \\ &= 4C(N/4)+N+N \\ &= 8C(N/8)+N+N+N \\ &= N \cdot 1 + \underbrace{N+N+\ldots+N}_{k=\lg N} \\ &= N + N\lg N \in \Theta(N \lg N) \end{aligned}$$

$(N \lg N)$ for arbitrary $N$ (not just $2^k$).

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## Using the Notation

order notation for any kind of real-valued function.

hem to describe cost functions. Example:

```
 position of X in list L, or -1 if not found. */
(List L, Object X) {

 = 0; L != null; L = L.next, c += 1)
 (X.equals(L.head)) return c;
 -1;
```

esentative operation: number of `.equals` tests.

th of $L$, then loop does *at most* $N$ tests: *worst-case* sts.

al # of instructions executed is roughly proportional worst case, so can also say worst-case time is $O(N)$, f units used to measure.

provision (in defn. of $O(\cdot)$) to handle empty list.

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## Effect of Nested Loops

s often lead to polynomial bounds:

```
 i = 0; i < A.length; i += 1)
nt j = 0; j < A.length; j += 1)
 (i != j && A[i] == A[j])
 return true;
lse;
```

is $O(N^2)$, where $N =$ `A.length`. *Worst-case time is*

icient though:

```
 i = 0; i < A.length; i += 1)
nt j = i+1; j < A.length; j += 1)
 (A[i] == A[j]) return true;
lse;
```

ase time is proportional to

$$-1+N-2+\ldots+1 = N(N-1)/2 \in \Theta(N^2)$$

ic time unchanged by the constant factor).

---

## Binary Search: Slow Growth

```
 is an element of S[L .. U]. Assumes
ding order, 0 <= L <= U-1 < S.length. */
tring X, String[] S, int L, int U) {
eturn false;
)/2;
 X.compareTo(S[M]);
 0) return isIn(X, S, L, M-1);
ect > 0) return isIn(X, S, M+1, U);
 true;
```

-case time, $C(D)$, (as measured by # of string compards on size $D = U - L + 1$.

`S[M]` from consideration each time and look at half the $D = 2^k - 1$ for simplicity, so:

$$\begin{aligned} C(D) &= \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases} \\ &= \underbrace{1+1+\ldots+1}_{k}+0 \\ &= k = \lceil \lg D \rceil \in \Theta(\lg D) \end{aligned}$$