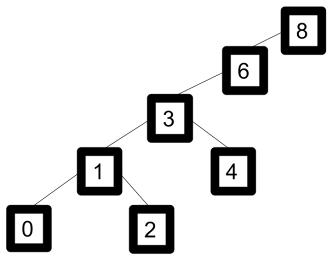
CS 61B
Fall 2021

# Balanced Search

Exam Prep Discussion 12: November 8, 2021

## 1 Balancing Trees

We are given the following extremely unbalanced search tree.




Select the minimum number of rotations in the correct order required to balance this tree. *Hint*: The resulting tree should have two layers of nodes below the root.

[ ] Rotate left on 8
[ ] Rotate right on 8
[ ] Rotate left on 6
[ ] Rotate right on 6
[ ] Rotate left on 4
[ ] Rotate right on 4
[ ] Rotate left on 3
[ ] Rotate right on 3
[ ] Rotate left on 2
[ ] Rotate right on 2
[ ] Rotate left on 1
[ ] Rotate right on 1
[ ] Rotate left on 0
[ ] Rotate right on 0

**Solution:**
[ ] Rotate left on 8
[X] Rotate right on 8
[ ] Rotate left on 6
[X] Rotate right on 6
[ ] Rotate left on 4
[ ] Rotate right on 4
[ ] Rotate left on 3
[ ] Rotate right on 3
[ ] Rotate left on 2
[ ] Rotate right on 2
[ ] Rotate left on 1

[ ] Rotate right on 1
[ ] Rotate left on 0
[ ] Rotate right on 0

**Explanation:** Rotating right on 8, then on 6, makes 3 the new root of the tree (with 6 as the right child). Verify this for yourself.

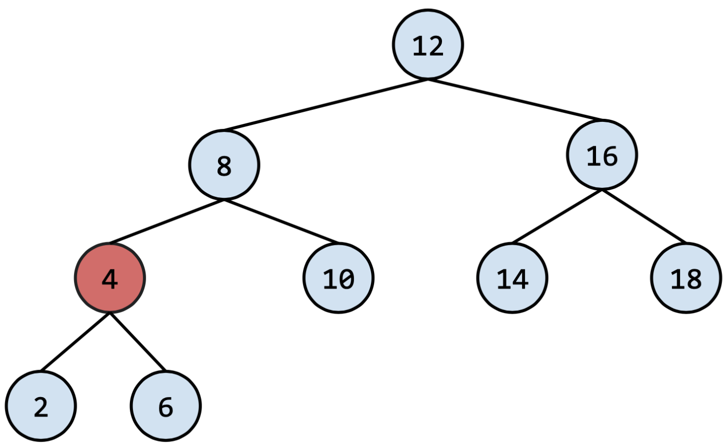
# 2 LLRBs

**a) (2 Points).** Perform the following insertions on the Left Leaning Red Black Tree (LLRB) given below. For each insertion, give the fix up operations needed. Recall a fix up operation is one of the following:

- `rotateLeft`
- `rotateRight`
- `colorFlip`
- change the root node to black.

Note that insertions are **dependent**. If only two operations are necessary, pick "None" for the third operation. If only one operation is necessary, pick "None" for the second and third operation. If no operations are necessary, pick "None" for all three operations.

If you put "None" for the "Operation applied", **leave the "Node to apply on" blank.** (Summer 2021 MT2)




**i) (0.5 Points).** Insert 17

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |

**Solution:**

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |

**Explanation:** 17 is inserted as the left child of 18. No fixes are required at this point.

**ii) (0.5 Points).** Insert 15. Note that insertions are dependent, so insert 15 into the state of the LLRB after the insertion of 17.

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |

**Solution:**

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | √ **`rotateLeft()`** ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | 14 |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |

**Explanation:** 15 is inserted as the right child of 14. This requires a left rotation of 14 to maintain the left-leaning invariant.

**iii) (0.75 Points).** Insert 13. Note that insertions are dependent, so insert 13 into the state of the LLRB after the insertion of 15.

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |

**Solution:**

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` √ **`rotateRight()`** ◯ `colorFlip()` ◯ change root to black ◯ `None` | 15 |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` √ **`colorFlip()`** ◯ change root to black ◯ `None` | 14 |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black √ **`None`** | |

**Explanation:** 13 is inserted as the left child of 14. This requires a right rotation on 15, since you cannot have 2 left red nodes in a row; then you must color flip 14 to break up the 4-node.

**iv) (0.75 Points).** Insert 19. Note that insertions are dependent, so insert 19 into the state of the LLRB after the insertion of 13.

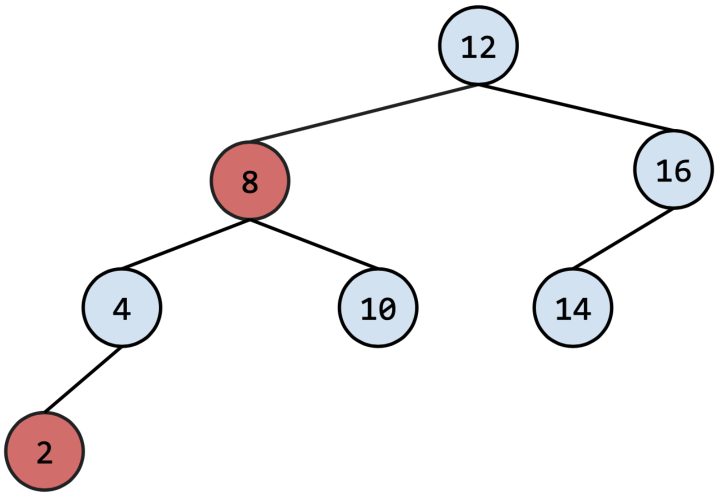
| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |
| 3rd operation | ◯ `rotateLeft()` ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | |

**Solution:**

| | Operation applied | Node to apply on |
|---|---|---|
| 1st operation | ◯ `rotateLeft()` ◯ `rotateRight()` √ **`colorFlip()`** ◯ change root to black ◯ `None` | 18 |
| 2nd operation | ◯ `rotateLeft()` ◯ `rotateRight()` √ **`colorFlip()`** ◯ change root to black ◯ `None` | 16 |
| 3rd operation | √ **`rotateLeft()`** ◯ `rotateRight()` ◯ `colorFlip()` ◯ change root to black ◯ `None` | 12 |

**Explanation:** 19 is inserted as the right child of 18. This requires a color flip on 18 to break up the 4-node, then a color flip on 16 which not has 2 red children. After this, a left rotation on 12 is required since it has a red right child.

**b) (1.5 Points).** The tree below is **not** a valid LLRB (hint: to see why this is the case, draw the corresponding 2-3 tree) but it's close! In this part, we will try to *transform* it into a valid LLRB in two different ways. Note that each way acts **independently** of the previous. If a way isn't possible, put `impossible`. Recall that LLRBs **cannot** have duplicates.




**i) (0.75 Points).** Way 1: Remove a **single leaf** node from the tree. Which leaf node?

◯ 2 ◯ 4 ◯ 8 ◯ 10 ◯ 12 ◯ 14 ◯ 16 ◯ impossible

**Solution:**

◯ 2 ◯ 4 ◯ 8 ◯ 10 ◯ 12 √ **14** ◯ 16 ◯ impossible

**Explanation:** A LLRB always has the same "black height" (number of black nodes from root to leaf). Note that the left child has a "black height" of 2 but the right has a black height of 3; thus deleting 14 makes this a valid LLRB.

**ii) (0.75 Points).** Way 2: Flip the color of a **single** node. Which node?

◯ 2 ◯ 4 ◯ 8 ◯ 10 ◯ 12 ◯ 14 ◯ 16 ◯ impossible

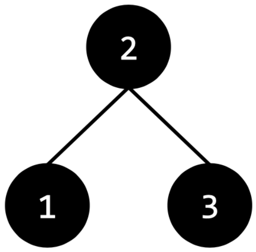
**Solution:**

◯ 2 ◯ 4 ◯ 8 ◯ 10 ◯ 12 √ **14** ◯ 16 ◯ impossible

**Explanation:** Like above, flipping 14 decreases the black height of the right child by 1, making it valid.

# 3 Trees

The simple tree below can be a BST, 2-3 Tree, or even an LLRB!




**a) (1 Point).** Suppose it is a BST. Select all the insertion orderings that can produce the BST above. (Summer 2021 MT2)

☐ 1, 2, 3 ☐ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Solution:**
☐ 1, 2, 3 ☐ 1, 3, 2 ■ 2, 1, 3 ■ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Explanation:** For 2 to be the root, it must be inserted first (otherwise the BST will be a linear chain). This corresponds to the third and fourth options.

**b) (1 Point).** Now, suppose it is a 2-3 Tree. Select all the insertion orderings that can produce the 2-3 Tree above.

☐ 1, 2, 3 ☐ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Solution:**
■ 1, 2, 3 ■ 1, 3, 2 ■ 2, 1, 3 ■ 2, 3, 1 ■ 3, 1, 2 ■ 3, 2, 1
☐ None of the above

**Explanation:** A 2-3 tree is always balanced, so any insertion order will result in the balanced binary tree above.

**c) (2.5 Points).** Now, suppose it is an LLRB with only black nodes.
**i) (0.75 Points).** Select all the insertion orderings that can produce the LLRB above.

☐ 1, 2, 3 ☐ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Solution:**
■ 1, 2, 3 ■ 1, 3, 2 ■ 2, 1, 3 ■ 2, 3, 1 ■ 3, 1, 2 ■ 3, 2, 1
☐ None of the above
**Explanation:** A LLRB with only black nodes is always balanced, so any insertion order will result in the balanced binary tree above.

**ii (0.75 Points).** Which insertion ordering requires the **minimum** number of `rotateLeft` and `rotateRight` calls. If multiple produce the minimum, select all.

☐ 1, 2, 3 ☐ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Solution:**
☐ 1, 2, 3 ☐ 1, 3, 2 ■ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above
**Explanation:** Note that if 2 is not inserted first, there will always be at least 1 rotation required to make it the root. If 3 is inserted before 1, there will be one rotation after 3 is inserted (2 cannot have a right red child), then another rotation to balance the tree after 1 is inserted. Thus, the optimal ordering is 213.

**iii) (1 Point).** Which insertion ordering requires **the maximum** number of `rotateLeft` and `rotateRight` calls. If multiple produce the maximum, select all.
☐ 1, 2, 3 ☐ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above

**Solution:**
☐ 1, 2, 3 ■ 1, 3, 2 ☐ 2, 1, 3 ☐ 2, 3, 1 ☐ 3, 1, 2 ☐ 3, 2, 1
☐ None of the above
**Explanation:** To maxmize the number of rotations, both insertions should be a right child (to force a left rotation). This only happens in the ordering 1, 3, 2. In particular, 1, 3, 2 requires 3 rotations (rotate 1 left, rotate 2 left, rotate 3 right).