CS 61B
Fall 2021

More Asymptotics

Exam Prep Discussion 8: October 11, 2021

1 Asymptotics is Fun!

(a) Using the function `g` defined below, what is the runtime of the following function calls? Write each answer in terms of `N`.

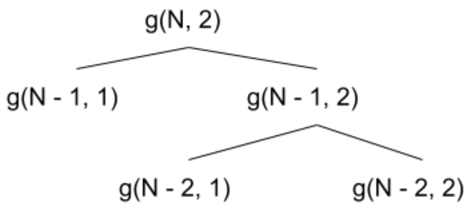
```
1 void g(int N, int x) {
2     if (N == 0) {
3         return;
4     }
5     for (int i = 1; i <= x; i++) {
6         g(N - 1, i);
7     }
8 }
```

`g(N, 1)`: $\Theta($ $)$

`g(N, 2)`: $\Theta($ $)$

Solution:
`g(N, 1)`: $\Theta(N)$
Explanation: When `x` is 1, the loop gets executed once and makes a single recursive call to `g(N - 1)`. The recursion goes `g(N)`, `g(N - 1)`, `g(N - 2)`, and so on. This is a total of N recursive calls, each doing constant work.

`g(N, 2)`: $\Theta(N^2)$
Explanation: When `x` is 2, the loop gets executed twice. This means a call to `g(N)` makes 2 recursive calls to `g(N - 1, 1)` and `g(N - 1, 2)`. The recursion tree looks like this:



From the first part, we know `g(..., 1)` does linear work. Thus, this is a recursion tree with N levels, and the total work is $(N-1)+(N-2)+...+1 = \Theta(N^2)$ work.

(b) Suppose we change line 6 to `g(N - 1, x)` and change the stopping condition in the for loop to `i <= f(x)` where `f` returns a random number between 1 and `x`, inclusive. For the following function calls, find the tightest Ω and big O bounds.

```
void g(int N, int x) {
    if (N == 0) {
        return;
    }
    for (int i = 1; i <= f(x); i++) {
        g(N - 1, x);
    }
}
```

`g(N, 2)`: Ω(   ), O(   )

`g(N, N)`: Ω(   ), O(   )

Solution:

`g(N, 2)`: $\Omega(N)$, $O(2^N)$

`g(N, N)`: $\Omega(N)$, $O(N^N)$

Explanation: Suppose `f(x)` always returns 1. Then, this is the same as case 1 from (a), resulting in a linear runtime.

On the other hand, suppose `f(x)` always returns x. Then `g(N, x)` makes `x` recursive calls to `g(N - 1, x)`, each of which makes `x` recursive calls to `g(N - 2, x)`, and so on, so the recursion tree has 1, x, x^2 ... nodes per level. Outside of the recursion, the function `g` does `x` work per node. Thus, the overall work is $x * 1 + x * x + x * x^2 + ... + x * x^{N-1} = x(1 + x + x^2 + ... + x^{N-1})$.

Plug in `x = 2` to get $2(1 + 2 + 2^2 + ... + 2^{N-1}) = O(2^N)$ for our first upper bound. Plug in `x = N` to get $N(1 + N + N^2 + ... + N^{N-1}) = O(N^N)$ (ignoring lower-order terms).

2 Flip Flop

Suppose we have the `flip` function as defined below. Assume the method `unknown` returns a random `integer` between `1` and `N`, exclusive, and runs in constant time. For each definition of the `flop` method below, give the best and worst case runtime of `flip` in $\Theta(.)$ notation as a function of `N`.

```
public static void flip(int N) {
    if (N <= 100) {
        return;
    }
    int stop = unknown(N);
    for (int i = 1; i < N; i++) {
        if (i == stop) {
            flop(i, N);
            return;
        }
    }
}
```

(a)

```
public static void flop(int i, int N) {
    flip(N - i);
}
```

Best Case: $\Theta(\quad)$, Worst Case: $\Theta(\quad)$

Solution:

Best Case: $\Theta(N)$, Worst Case: $\Theta(N)$

Explanation: Consider some arbitrary value of `stop`. When `stop = x`, we do `x` work inside of `flip` (the `for` loop) and recursively call `flip(N - x)` through `flop`. This results in a total of `N / x` calls before reaching our base case, and `x` work per call, for a total of $\Theta(N)$ work. Note that this holds for any value of `x`, so our best and worst case are the same.

(b)

```
public static void flop(int i, int N) {
    int minimum = Math.min(i, N - i);
    flip(minimum);
    flip(minimum);
}
```

Best Case: $\Theta(\quad)$, Worst Case: $\Theta(\quad)$

Solution:

Best Case: $\Theta(1)$, Worst Case: $\Theta(N\log(N))$

Explanation: In the best case, `stop = 1`. This hits the base case immediately, so we make 2 calls to `flip` then stop for $\Theta(1)$ work.

In the worst case, `stop = N / 2`. This results in `flip` making 2 recursive calls to itself with the argument `N / 2`. Note the similarity of this recurrence and mergesort; the runtime is the same $\Theta(N\log N)$.

(c)
```
public static void flop(int i, int N) {
    flip(i);
    flip(N - i);
}
```

Best Case: $\Theta($ $)$, Worst Case: $\Theta($ $)$

Solution:

Best Case: $\Theta(N)$, Worst Case: $\Theta(N^2)$

Explanation: In the best case, suppose `stop = 1`. Then `flip(N)` makes recursive calls to `flip(1)` and `flip(N - 1)`, the first of which terminates immediately in the base case. `flip(N - 1)` then calls `flip(1)` and `flip(N - 2)`. The pattern is a linear recursion: constant work per call, N calls total for $\Theta(N)$ work.

In the worst case, suppose `stop = N - 1`. Note that this case is symmetrical to the best case in terms of recursive calls; however we do work proportional to `N` inside of `flip` each time because of the `for` loop. The overall work is $(N-1)+(N-2)+(N-3)+...+2+1=\Theta(N^2)$.

3 Prime Factors

Determine the best and worst case runtime of `prime_factors` in $\Theta(.)$ notation as a function of `N`.

```java
int prime_factors(int N) {
    int factor = 2;
    int count = 0;
    while (factor * factor <= N) {
        while (N % factor == 0) {
            System.out.println(factor);
            count += 1;
            N = N / factor;
        }
        factor += 1;
    }
    return count;
}
```

Best Case: $\Theta(\quad)$, Worst Case: $\Theta(\quad)$

Solution:
Best Case: $\Theta(log(N))$, Worst Case: $\Theta(\sqrt{N})$
Explanation: In the best case, `N` is some power of 2. Then the inner while loop will halve `N` each time until it becomes 1. At this point, both the inner and outer while loop conditions will be false and the function will return. Halving `N` each time results in a $\Theta(\log N)$ runtime.
In the worst case, `N` will not be divisible by any value of `factor`. This means we increment `factor` by 1 each time until `factor * factor > N`. This is at most $\sqrt{N}$ loops.