CS 61B Spring 2022 — Disjoint Sets, Graph Algorithms

Exam Prep Discussion 14: April 11, 2022

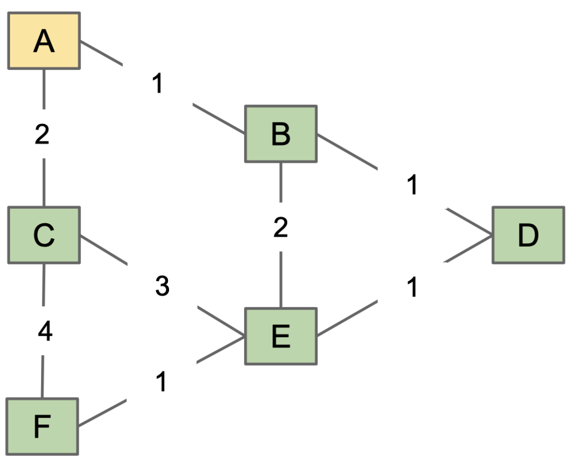
1 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression and explain your reasoning.

```
      i:   0   1   2   3   4   5   6   7   8   9
      ----------------------------------------
A. a[i]:   1   2   3   0   1   1   1   4   4   5
B. a[i]:   9   0   0   0   0   0   9   9   9 -10
C. a[i]:   1   2   3   4   5   6   7   8   9 -10
D. a[i]: -10   0   0   0   0   1   1   1   6   2
E. a[i]: -10   0   0   0   0   1   1   1   6   8
F. a[i]:  -7   0   0   1   1   3   3  -3   7   7
```

2 Graph Algorithms

a) (4 Points). Suppose we have the graph below.



For all parts, break ties alphabetically. We recommend drawing this graph on paper! Recall that SPT stands for shortest paths tree.

i) (1 Points). Run Dijkstra's from vertex `A` on the graph above.

1. **(0.5 Points).** What is the ordering that vertices are visited? Format your answer as a space separated list, e.g. `A B C D E F`.

   Order:

2. **(0.5 Points).** What edges are included in the SPT rooted at A?

☐ AB ☐ AC ☐ BD ☐ BE ☐ CE ☐ CF ☐ DE ☐ EF

ii) (1 Points). Run A* from vertex `A` to `F` on the graph above. Let the heuristic of a vertex `v` be the number of edges between `v` and `F` in the BFS tree rooted at `F` (you may need to run BFS to find this). For instance, the heuristic of D is 2.

1. **(0.5 Points).** What is the ordering that vertices are visited? Format your answer as a space separated list, e.g. `A B C D E F`.

   Order:

2. **(0.5 Points).** What edges comprise the shortest path?

   ☐ AB ☐ AC ☐ BD ☐ BE ☐ CE ☐ CF ☐ DE ☐ EF

iii) (1 Points). Run Prim's from vertex `A` on the graph above.

1. **(0.5 Points).** What is the ordering that vertices are visited? Format your answer as a space separated list, e.g. `A B C D E F`.

   Order:

2. **(0.5 Points).** What edges are included in the MST rooted at A?

   ☐ AB ☐ AC ☐ BD ☐ BE ☐ CE ☐ CF ☐ DE ☐ EF

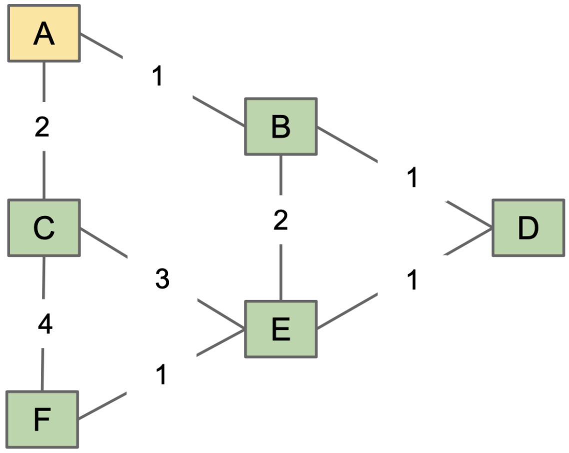
iv) (0.5 Points). Increase the weight of one edge by 1 so that the SPT found by Dijkstra's is the same as the MST found by Prim's. Assume we've run Prim's and Dijkstra's from vertex `A` on the graph above.

If the SPT found by Dijkstra's is already the same as the MST found by Prim's, select the option "no change needed". If it's impossible for them to be equal after increasing the weight of one edge by 1, select the option "impossible".

◯ AB ◯ AC ◯ BD ◯ BE ◯ CE ◯ CF ◯ DE ◯ EF
◯ no change needed ◯ impossible

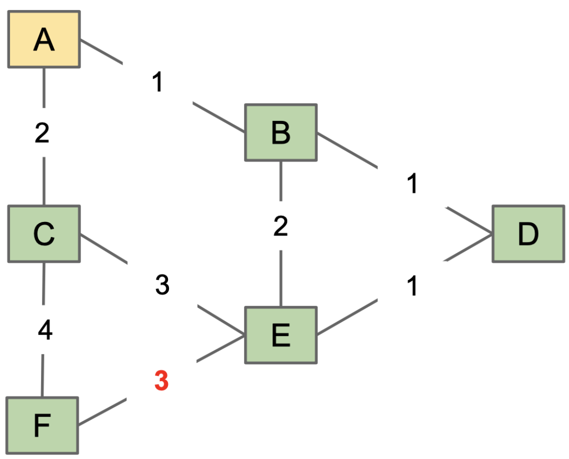
b) (2 Points). Multiple SPTs

i) (0.5 Points). Using the graph from the previous part, which has been recopied for your convenience below, how many different **SPTs** rooted at A exist? Recall there can be multiple SPTs for a given start vertex if there are multiple paths of equal, minimum cost to at least one end vertex.



Number of SPTs: ◯ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6 ◯ 7 ◯ 8

ii) (0.5 Points). Now, suppose we change the edge weight of `EF` to 3. How many SPTs are there now? The altered graph has been drawn for your convenience below. Hint: The number of SPTs should *increase* from part i to part ii.



Number of SPTs: ◯ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6 ◯ 7 ◯ 8

iii) (1 Point). Finally, using the same graph from part b.ii, change *one* edge weight to further **increase** the total number of SPTs rooted at A.

Edge: ◯ AB ◯ AC ◯ BD ◯ BE ◯ CE ◯ CF ◯ DE ◯ EF

New weight: ◯ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6 ◯ 7 ◯ 8

3 Multiple MSTs

Recall a graph can have multiple MSTs if there are multiple spanning trees of minimum weight.

(a) For each subpart below, select the correct option and justify your answer. If you select "never" or "always," provide a short explanation. If you select "sometimes", provide two graphs that fulfill the given properties — one with multiple MSTs and one without. Assume G is an undirected, connected graph.

1. If **none** the edge weights are **identical**, there will

   ◯ never be multiple MSTs in G.

   ◯ sometimes be multiple MSTs in G.

   ◯ always be multiple MSTs in G.

   Justification:

2. If **some** of the edge weights are **identical**, there will

   ◯ never be multiple MSTs in G.

   ◯ sometimes be multiple MSTs in G.

   ◯ always be multiple MSTs in G.

   Justification:

3. If **all** of the edge weights are **identical**, there will

   ◯ never be multiple MSTs in G.

   ◯ sometimes be multiple MSTs in G.

   ◯ always be multiple MSTs in G.

   Justification:

(b) Suppose we have a connected, undirected graph G with N vertices and N edges, where all the **edge weights are identical**. Find the maximum and minimum number of MSTs in G and explain your reasoning.

Minimum: _________
Maximum: _________

Justification:

(c) It is possible that Prim's and Kruskal's find **different** MSTs on the same graph `G` (as an added exercise, construct a graph where this is the case!). Given any graph `G` with integer edge weights, modify G to **ensure** that Prim's and Kruskal's will always find the same MST. You may not modify Prim's or Kruskal's.
Hint: Look at subpart 1 of part a.