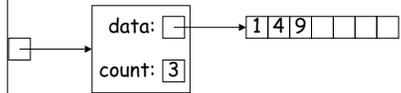




## Implementing Lists (I): ArrayLists

Concrete types in Java library for interface List are ArrayList and LinkedList:

As you expect, an ArrayList, A, uses an array to hold data. For example, a list containing the three items 1, 4, and 9 might be like this:



If you add four more items to A, its data array will be full, and the data will have to be replaced with a pointer to a new array, that starts with a copy of its previous values.

For best performance, how big should this new array be? Doubling the size by 1 each time it gets full (or by any constant factor) means the cost of  $N$  additions will scale as  $\Theta(N^2)$ , which makes ArrayList look much worse than LinkedList (which uses an amortized implementation.)

## Amortized Time

Let the actual costs of a sequence of  $N$  operations be  $c_1, \dots, c_N$ , which may differ from each other by arbitrary amounts. Let  $f(i)$  be the amortized cost of the  $i$ th operation.

We say the sequence is amortized if there is a sequence  $a_0, a_1, \dots, a_{N-1}$ , where  $a_i \in O(g(i))$ .

$$\sum_{0 \leq i < k} a_i \geq \sum_{0 \leq i < k} c_i \text{ for all } k,$$

then the operations all run in  $O(g(i))$  amortized time.

The amortized cost of a given operation,  $a_i$ , may be arbitrarily greater than or equal to the total actual time, but the sequence of operations stops—i.e., no matter what

we choose, the amortized time bounds are much less than the individual time bounds:  $g(i) \ll f(i)$ .

In the case of insertion with array doubling,  $f(i) \in O(N)$  and

## Calculating Amortized Time: Potential Method

For the argument, associate a *potential*,  $\Phi_i \geq 0$ , to the  $i$ th operation. It keeps track of "saved up" time from cheap operations that can be "spent" on later expensive ones. Start with  $\Phi_0 = 0$ .

We pretend that the cost of the  $i$ th operation is actually  $a_i$ , the amortized cost, defined

$$a_i = c_i + \Phi_{i+1} - \Phi_i,$$

where  $c_i$  is the real cost of the operation. Or, looking at potential:

$$\Phi_{i+1} = \Phi_i + (a_i - c_i)$$

For operations, we artificially set  $a_i > c_i$  so that we can increase the potential ( $\Phi_{i+1} > \Phi_i$ ).

For expensive ones, we typically have  $a_i \ll c_i$  and greatly decrease  $\Phi$  so that it goes negative—may not be "overdrawn".

To make all this so that  $a_i$  remains as we desired (e.g.,  $O(1)$  for array), without allowing  $\Phi_i < 0$ .

If we choose  $a_i$  so that  $\Phi_i$  always stays ahead of  $c_i$ .

## The List Interface

Section

represent *indexed sequences* (generalized arrays)

Methods to those of Collection:

Utility methods: indexOf, lastIndexOf.

Methods: get(i), listIterator(), sublist(B, E).

add and addAll with additional index to say *where* to add. remove and removeAll for removal operations. set operation to go with

Iterator<Item> extends Iterator<Item>:

hasNext and hasPrevious.

add, remove, and set allow one to iterate through a list, inserting, removing, or changing as you go.

**Question:** What advantage is there to saying List L is an ArrayList L or LinkedList L?

## Expanding Vectors Efficiently

For an array for expanding sequence, best to *double* the size of the array. Here's why.

If you double the size  $s$ , doubling its size and moving  $s$  elements to the new array is time proportional to  $2s$ .

There is an additional  $\Theta(1)$  cost for each addition to the array, so the total cost is actually assigning the new value into the array.

Adding up these costs for inserting a sequence of  $N$  items, it turns out to be proportional to  $N$ , as if each addition took constant time, even though some of the additions actually take time proportional to  $N$  all by themselves!

## Amortization: Expanding Vectors (II)

	Resizing Cost	Cumulative Cost	Resizing Cost per Item	Array Size After Insertions
	0	0	0	1
	2	2	1	2
	4	6	2	4
	0	6	1.5	4
	8	14	2.8	8
	0	14	2.33	8
	:	:	:	:
	0	14	1.75	8
	16	30	3.33	16
	:	:	:	:
	0	30	1.88	16
	:	:	:	:
	:	:	:	:
-1	0	$2^{m+2} - 2$	$\approx 2$	$2^{m+1}$
	$2^{m+2}$	$2^{m+3} - 2$	$\approx 4$	$2^{m+2}$

Without amortizing the cost of resizing, we average at most  $2$  units for resizing on each item: "amortized resizing cost." Time to add  $N$  elements is  $\Theta(N)$ , *not*  $\Theta(N^2)$ .

## Application to Expanding Arrays

When adding element  $i$  to our array, the cost,  $c_i$ , of adding element  $i$  when there is already space for it is 1 unit.

Otherwise, we do not initially have space when adding items 1, 2, 4, 8, ... In other words at item  $2^n$  for all  $n \geq 0$ . So,

$c_i \geq 0$  and is not a power of 2; and

When  $i$  is a power of 2 (copy  $i$  items, clear another  $i$  items, then add item  $i$ ).

At operation  $2^n$  we're going to need to have saved up at least  $2^{n+1}$  units of potential to cover the expense of expanding the array. And we have this operation and the preceding  $2^n - 1$  operations, which to save up this much potential (everything since the last doubling operation).

$c_i = 1$  and  $a_i = 5$  for  $i > 0$ . Apply  $\Phi_{i+1} = \Phi_i + (a_i - c_i)$ , and

happens:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	<i>Pretending each cost is 5 never underestimates true cumulative time.</i>
9	1	1	1	17	1	1	1	1	1	1	1	33	1	
5	5	5	5	5	5	5	5	5	5	5	5	5	5	
6	2	6	10	14	2	6	10	14	18	22	26	30	2	