CS61B Lecture #20: Trees

A Recursive Structure

- Trees naturally represent recursively defined, hierarchical objects with more than one recursive subpart for each instance.
- Common examples: expressions, sentences.
  - Expressions have definitions such as "an expression consists of a literal or two expressions separated by an operator."
- Also describe search structures in which we recursively divide a set into multiple disjoint subsets.

Formal Definitions

- Trees come in a variety of flavors, all defined recursively:
  - **61A style:** A tree consists of a *label* value and zero or more *branches* (or *children*), each of them a tree.
  - **61A style, alternative definition:** A tree is a set of *nodes* (or *vertices*), each of which has a label value and one or more *child nodes*, such that no node descends (directly or indirectly) from itself. A node is the *parent* of its children.
  - **Positional trees:** A tree is either *empty* or consists of a node containing a label value and an indexed sequence of zero or more children, each a positional tree. If every node has two positions, we have a *binary tree* and the children are its *left and right subtrees.* Again, nodes are the parents of their non-empty children.
  - We'll see other varieties when considering graphs.

Tree Characteristics (I)

- The *root* of a tree is a non-empty node with no parent in that tree (its parent might be in some larger tree that contains that tree as a subtree). Thus, every node is the root of a (sub)tree.
- The *order*, *arity*, or *degree* of a node (tree) is its number (maximum number) of children.
- The nodes of a *k-ary tree* each have at most k children.
- A *leaf* node has no children (no non-empty children in the case of positional trees).

Tree Characteristics (II)

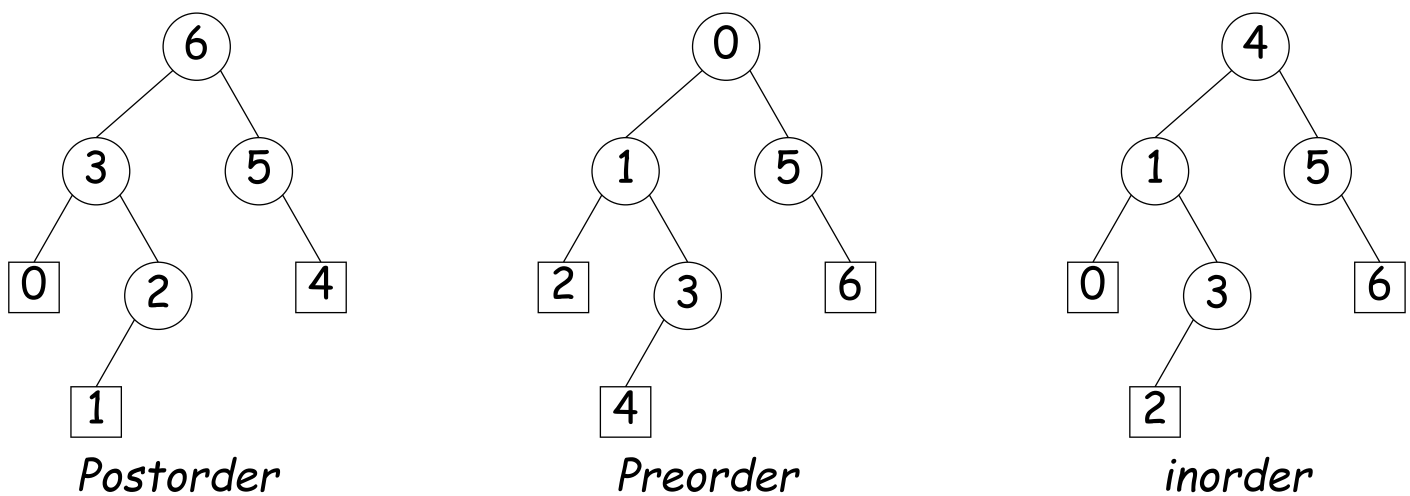
- The *height* of a node in a tree is the largest distance to a leaf. That is, a leaf has height 0 and a non-empty tree's height is one more than the maximum height of its children. The height of a tree is the height of its root.
- The *depth* of a node in a tree is the distance to the root of that tree. That is, in a tree whose root is R, R itself has depth 0 in R, and if node $S \neq R$ is in the tree with root R, then its depth is one greater than its parent's.

A Tree Type, 61A Style

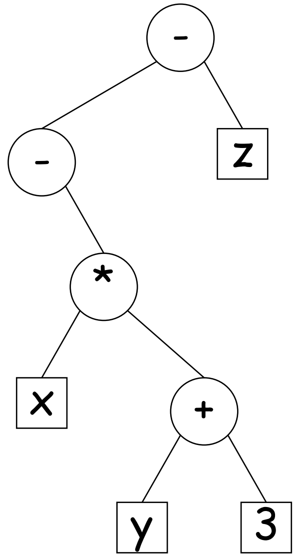
```
/** An immutable Tree whose labels are of type LABEL. */
public class Tree<Label> {

    @SuppressWarnings("unchecked")  // Explained in a later lecture
    public Tree(Label label, Tree<Label>... children) {
        _label = label;
        _kids = new ArrayList<>(Arrays.asList(children));
    }

    public int arity() { return _kids.size(); }

    public Label label() { return _label; }

    public Tree<Label> child(int k) { return _kids.get(k); }

    private Label _label;
    private ArrayList<Tree<Label>> _kids;
}
```

Fundamental Operation: Traversal

- *Traversing a tree* means enumerating (some subset of) its nodes.
- Typically done recursively, because that is natural description.
- As nodes are enumerated, we say they are *visited*.
- Three basic orders for enumeration (+ variations):
  - **Preorder:** visit node, traverse its children.
  - **Postorder:** traverse children, visit node.
  - **Inorder:** traverse first child, visit node, traverse second child (binary trees only).



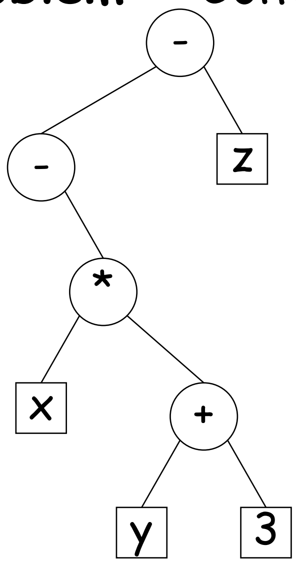
Preorder Traversal and Prefix Expressions

Problem: Convert  **into** `(- (- (* x (+ y 3))) z)`


```
static String toLisp(Tree<String> T) {
  if (T.arity() == 0) return T.label();
  else {
    String R;  R = "(" + T.label();
    for (int i = 0; i < T.arity(); i += 1)
      R += " " + toLisp(T.child(i));
    return R + ")";
  }
}
```

Inorder Traversal and Infix Expressions

Problem: Convert



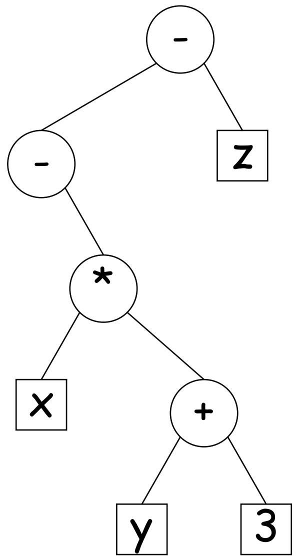
into ((-(x*(y+3)))-z)

To think about: how to get rid of all those parentheses.

```
static String toInfix(Tree<String> T) {
  if (T.arity() == 0) {
    return T.label();
  } else if (T.arity() == 1) {
    return "(" T.label() + toInfix(T.child(0)) + ")";
  } else {
    return "(" toInfix(T.child(0)) + T.label() + toInfix(T.child(1)) + ")";
  }
}
```

Postorder Traversal and Postfix Expressions

Problem: Convert



into `x y 3 +:2 *:2 -:1 z -:2`

```
static String toPolish(Tree<String> T) {
  String R; R = "";
  for (int i = 0; i < T.arity(); i += 1)
     R += toPolish(T.child(i)) + " ";
  return R + String.format("%s:%d", T.label(), T.arity());
}
```

A General Traversal: The Visitor Pattern

```
void preorderTraverse(Tree<Label> T, Consumer<Tree<Label>> visit)
{
  if (T != null) {
    visit.accept(T);
    for (int i = 0; i < T.arity(); i += 1)
      preorderTraverse(T.child(i), visit);
  }
}
```

- `java.util.function.Consumer<AType>` is a library interface that works as a function-like type with one void method, `accept`, which takes an argument of type `AType`.
- Now, using Java 8 lambda syntax, I can print all labels in the tree in preorder with:

```
preorderTraverse(myTree, T -> System.out.print(T.label() + " "));
```

Iterative Depth-First Traversals

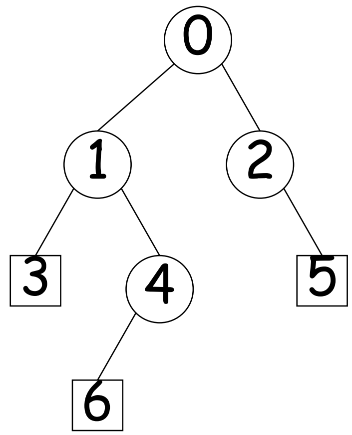
- Tree recursion conceals data: a *stack* of nodes (all the `T` arguments) and a little extra information. Can make the data explicit:

```
void preorderTraverse2(Tree<Label> T, Consumer<Tree<Label>> visit) {
  Stack<Tree<Label>> work = new Stack<>();
  work.push(T);
  while (!work.isEmpty()) {
    Tree<Label> node = work.pop();
    visit.accept(node);
    for (int i = node.arity()-1; i >= 0; i -= 1)
        work.push(node.child(i));  // Why backward?
  }
}
```

- This traversal takes the same $\Theta(\cdot)$ time as doing it recursively, and also the same $\Theta(\cdot)$ space.
- That is, we have substituted an explicit stack data structure (`work`) for Java's built-in execution stack (which handles function calls).

Level-Order (Breadth-First) Traversal

Problem: Traverse all nodes at depth 0, then depth 1, etc:



Breadth-First Traversal Implemented

A simple modification to iterative depth-first traversal gives breadth-first traversal. Just change the (LIFO) stack to a (FIFO) queue:

```
void breadthFirstTraverse(Tree<Label> T, Consumer<Tree<Label>> visit) {
  ArrayDeque<Tree<Label>> work = new ArrayDeque<>();  // (Changed)
  work.push(T);
  while (!work.isEmpty()) {
    Tree<Label> node = work.remove();     // (Changed)
    if (node != null) {
        visit.accept(node);
        for (int i = 0; i < node.arity(); i += 1) // (Changed)
           work.push(node.child(i));
    }
  }
}
```

Times

- The traversal algorithms have roughly the form of the `boom` example in §1.3.3 of *Data Structures*—an exponential algorithm.
- However, the role of M in that algorithm is played by the *height* of the tree, not the number of nodes.
- In fact, easy to see that tree traversal is *linear:* $\Theta(N)$, where N is the # of nodes: Form of the algorithm implies that there is one visit at the root, and then one visit for every *edge* in the tree. Since every node but the root has exactly one parent, and the root has none, must be $N-1$ edges in any non-empty tree.
- In positional tree, is also one recursive call for each empty tree, but # of empty trees can be no greater than kN, where k is arity.
- For k-ary tree (max # children is k), $h+1 \leq N \leq \frac{k^{h+1}-1}{k-1}$, where h is height.
- So $h \in \Omega(\log_k N) = \Omega(\lg N)$ and $h \in O(N)$.
- Many tree algorithms look at one child only. For them, worst-case time is proportional to the *height* of the tree—$\Theta(\lg N)$—assuming that tree is *bushy*—each level has about as many nodes as possible.

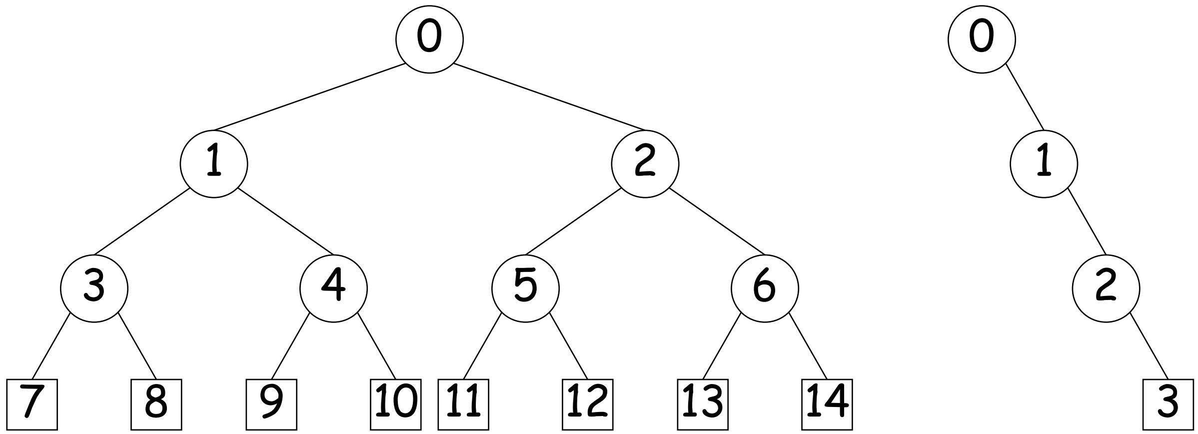
Recursive Breadth-First Traversal: Iterative Deepening

- Previous breadth-first traversal used space proportional to the *width* of the tree, which is $\Theta(N)$ for bushy trees, whereas depth-first traversal takes $\lg N$ space on bushy trees.
- Can we get breadth-first traversal in $\lg N$ space and $\Theta(N)$ time on bushy trees?
- For each level, k, of the tree from 0 to lev, call `doLevel(T,k)`:

```
void doLevel(Tree T, int lev) {
  if (lev == 0)
    visit T
  else
    for each non-null child, C, of T {
      doLevel(C, lev-1);
    }
}
```

- So we do breadth-first traversal by repeated (truncated) depth-first traversals: *iterative deepening*.
- In `doLevel(T, k)`, we skip (i.e., traverse but don't visit) the nodes before level k, and then visit at level k, but not their children.

Iterative Deepening Time?



- Let h be height, N be # of nodes.
- Count # edges traversed (i.e, # of calls, not counting null nodes).
- First (full) tree: 1 for level 0, 3 for level 1, 7 for level 2, 15 for level 3.
- Or in general $(2^1-1)+(2^2-1)+\ldots+(2^{h+1}-1) = 2^{h+2}-2-(h+1) \in \Theta(N)$, since $N = 2^{h+1} - 1$ for this tree.
- Second (*right leaning*) tree: 1 for level 0, 2 for level 2, 3 for level 3.
- Or in general $(h+1)(h+2)/2 = N(N+1)/2 \in \Theta(N^2)$, since $N = h+1$ for this kind of tree.

Iterators for Trees

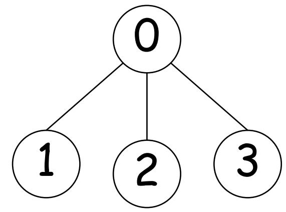
- Frankly, iterators are not terribly convenient on trees.
- But can use ideas from iterative methods.

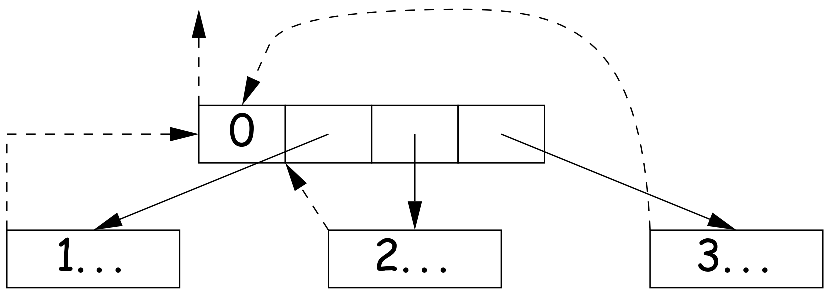
```
class PreorderTreeIterator<Label> implements Iterator<Label> {
  private Stack<Tree<Label>> s = new Stack<Tree<Label>>();

  public PreorderTreeIterator(Tree<Label> T) { s.push(T);  }

  public boolean hasNext() { return !s.isEmpty(); }
  public T next() {
    Tree<Label> result = s.pop();
    for (int i = result.arity()-1; i >= 0; i -= 1)
      s.push(result.child(i));
    return result.label();
  }
}
```

Example: (what do I have to add to class `Tree` first?)

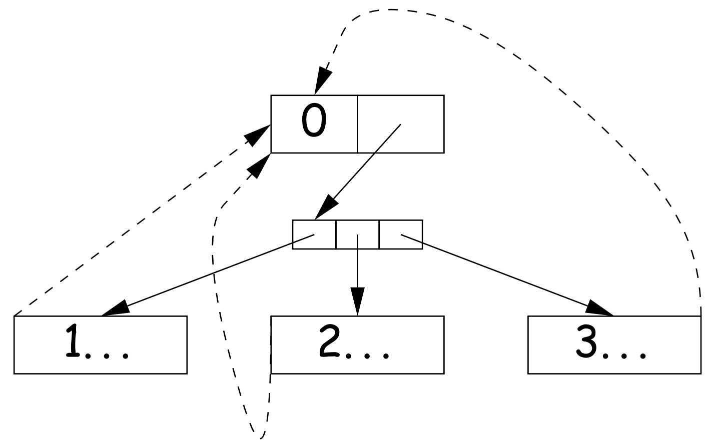
```
for (String label : aTree) System.out.print(label + " ");
```

Tree Representation

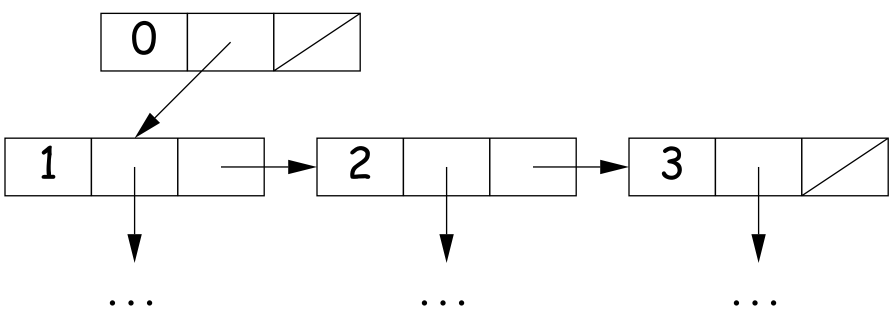




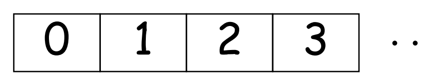
(a) Embedded child pointers
(+ optional parent pointers)



(b) Array of child pointers
(+ optional parent pointers)



(c) child/sibling pointers



(d) breadth-first array
(complete trees)