CS61B Lecture #26

Today:

- Sorting algorithms: why?
- Insertion Sort.
- Inversions
- Selection sorting.
- Heapsort.

Purposes of Sorting

- Sorting supports searching; binary search is a standard example.
- Also supports other kinds of search:
  - Are there two equal items in this set?
  - Are there two items in this set that both have the same value for property X?
  - What are my nearest neighbors?
- Used in numerous unexpected algorithms, such as convex hull (smallest convex polygon enclosing set of points).

Some Definitions

- A *sorting algorithm* (or *sort*) *permutes* (re-arranges) a sequence of elements to brings them into order, according to some *total order.*
- A total order, $\preceq$, is:
  - **Total:** $x \preceq y$ or $y \preceq x$ for all x, y.
  - **Reflexive:** $x \preceq x$;
  - **Antisymmetric:** $x \preceq y$ and $y \preceq x$ iff $x = y$.
  - **Transitive:** $x \preceq y$ and $y \preceq z$ implies $x \preceq z$.
- However, our orderings may treat unequal items as equivalent:
  - E.g., there can be two dictionary definitions for the same word. If we sort only by the word being defined (ignoring the definition), then sorting could put either entry first.
  - A sort that does not change the relative order of equivalent entries (compared to the input) is called *stable.*

Classifications

- *Internal sorts* keep all data in primary memory.
- *External sorts* process large amounts of data in batches, keeping what won't fit in secondary storage (in the old days, tapes).
- *Comparison-based* sorting assumes only thing we know about keys is their order.
- *Radix sorting* uses more information about key structure.
- *Insertion sorting* works by repeatedly inserting items at their appropriate positions in the sorted sequence being constructed.
- *Selection sorting* works by repeatedly selecting the next larger (smaller) item in order and adding it to one end of the sorted sequence being constructed.

Sorting Arrays of Primitive Types in the Java Library

- The java library provides static methods to sort arrays in the class `java.util.Arrays`.
- For each primitive type `P` other than `boolean`, there are

```
/** Sort all elements of ARR into non-descending order. */
static void sort(P[] arr) { ... }

/** Sort elements FIRST .. END-1 of ARR into non-descending
 *  order. */
static void sort(P[] arr, int first, int end) { ... }

/** Sort all elements of ARR into non-descending order,
 *  possibly using multiprocessing for speed. */
static void parallelSort(P[] arr) { ... }

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```

Sorting Arrays of Reference Types in the Java Library

- For reference types, `C`, that have a *natural order* (that is, that implement `java.lang.Comparable`), we have four analogous methods (one-argument `sort`, three-argument `sort`, and two `parallelSort` methods):

```
static <C extends Comparable<? super C>> void sort(C[] arr) {...}
static <C extends Comparable<? super C>>
    void sort(C[] arr, int first, int end) { ... }
static <C extends Comparable<? super C>>
    void parallelSort(P[] arr) { ... }
static <C extends Comparable<? super C>>
    void parallelSort(P[] arr, int first, int end) {...}
```

- And for all reference types, `R`, we have four more:

```
/** Sort all elements of ARR stably into non-descending order
 *  according to the ordering defined by COMP. */
static <R> void sort(R[] arr, Comparator<? super R> comp) {...}
```

and so forth.

- **Q:** Why the fancy generic arguments for `comp`?

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```

  and so forth.

- **Q:** Why the fancy generic arguments for `comp`?

  A: We want to allow types that have `compareTo` methods that apply also to more general types.

Sorting Lists in the Java Library

- The class `java.util.Collections` contains two methods similar to the sorting methods for arrays of reference types:

```
/** Sort all elements of LST stably into non-descending
 *  order. */
static <C extends Comparable<? super C>> sort(List<C> lst) {...}
```
etc.

```
/** Sort all elements of LST stably into non-descending
 *  order according to the ordering defined by COMP. */
static <R> void sort(List<R> , Comparator<? super R> comp) {...}
```
etc.

- Also a default instance method in the `List<R>` interface itself:

```
/** Sort all elements of LST stably into non-descending
 *  order according to the ordering defined by COMP. */
default void sort(Comparator<? super R> comp) {...}
```

Examples

- Assume:

```
import static java.util.Arrays.*;
import static java.util.Collections.*;
```

- Sort `X`, a `String[]` or `List<String>`, into non-descending order:

```
sort(X);    // or ...
```

- Sort `X` into reverse order (Java 8):

```
sort(X, (String x, String y) -> { return y.compareTo(x); });
// or
sort(X, Collections.reverseOrder());   // or
X.sort(Collections.reverseOrder());    // for X a List
```

- Sort `X[10], ..., X[100]` in array or `List X` (rest unchanged):

```
sort(X, 10, 101);
```

- Sort `L[10], ..., L[100]` in list `L` (rest unchanged):

```
sort(L.sublist(10, 101));
```

Sorting by Insertion

- Simple idea:
  - Starting with an empty sequence of outputs.
  - Add each item from the input, *inserting* it into the output sequence at the right point.
- Very simple, good for small sets of data.
- With vector or linked list, time for find + insert of one item is at worst $\Theta(k)$, where k is # of outputs so far.
- This gives us a $\Theta(N^2)$ algorithm (worst case as usual).
- Can we say more?

Inversions

- Can run in $\Theta(N)$ comparisons if already sorted.
- Consider a typical implementation for arrays:

```
for (int i = 1; i < A.length; i += 1) {
  int j;
  Object x = A[i];
  for (j = i-1; j >= 0; j -= 1) {
    if (A[j].compareTo(x) <= 0)  /* (1) */
      break;
    A[j+1] = A[j];               /* (2) */
  }
  A[j+1] = x;
}
```

- #times (1) executes for each `j` $\approx$ how far `x` must move.
- If each item is within K of its proper places, requires $O(KN)$ operations.
- Thus is good for any amount of *nearly sorted* data.
- One measure of unsortedness: # of *inversions:* pairs that are out of order (= 0 when sorted, $N(N-1)/2$ when reversed).
- Each execution of (2) decreases inversions by 1.

Shell's sort

Idea: Improve insertion sort by first sorting *distant* elements:

- First sort subsequences of elements $2^k - 1$ apart:
  - sort items #$0,\ 2^k - 1,\ 2(2^k - 1),\ 3(2^k - 1),\ \ldots$, then
  - sort items #$1,\ 1 + 2^k - 1,\ 1 + 2(2^k - 1),\ 1 + 3(2^k - 1),\ \ldots$, then
  - sort items #$2,\ 2 + 2^k - 1,\ 2 + 2(2^k - 1),\ 2 + 3(2^k - 1),\ \ldots$, then
  - *etc.*
  - sort items #$2^k - 2,\ 2(2^k - 1) - 1,\ 3(2^k - 1) - 1,\ \ldots$,
  - Each time an item moves, can reduce #inversions by as much as $2^{k+1} - 3$.
- Now sort subsequences of elements $2^{k-1} - 1$ apart:
  - sort items #$0,\ 2^{k-1} - 1,\ 2(2^{k-1} - 1),\ 3(2^{k-1} - 1),\ \ldots$, then
  - sort items #$1,\ 1 + 2^{k-1} - 1,\ 1 + 2(2^{k-1} - 1),\ 1 + 3(2^{k-1} - 1),\ \ldots$,
  - $\vdots$
- End at plain insertion sort ($2^0 = 1$ apart), but with most inversions gone.
- Sort is $\Theta(N^{3/2})$ (take CS170 for why!).

Example of Shell's Sort

																#I	#C
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	120	0
0	14	13	12	11	10	9	8	7	6	5	4	3	2	1	15	91	1
0	7	6	5	4	3	2	1	14	13	12	11	10	9	8	15	42	11
0	1	3	2	4	6	5	7	8	10	9	11	13	12	14	15	4	31
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	0	50

I: Inversions left.

C: Cumulative comparisons used to sort subsequences by insertion sort.

Sorting by Selection: Straight Selection

- **Idea:** Keep selecting the smallest (or largest) element, and adding it to the appropriate end of the result.
- Gives us *straight selection sort:*

```
<T extends Comparable<T>> void selectionSort(T[] A) {
    for (int i = A.length; i > 0; i -= 1) {
        int maxIndex;
        maxIndex = 0;
        for (int k = 1; k < i; k += 1) {
            if (A[maxIndex].compareTo(A[k]) < 0) {
                maxIndex = k;
            }
        }
        T temp = A[maxIndex]; A[maxIndex] = A[i - 1]; A[i - 1] = temp;
    }
}
```

- Time bound?

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        for (int k = 1; k < i; k += 1) {
            if (A[maxIndex].compareTo(A[k]) < 0) {
                maxIndex = k;
            }
        }
        T temp = A[maxIndex]; A[maxIndex] = A[i - 1]; A[i - 1] = temp;
    }
}
```
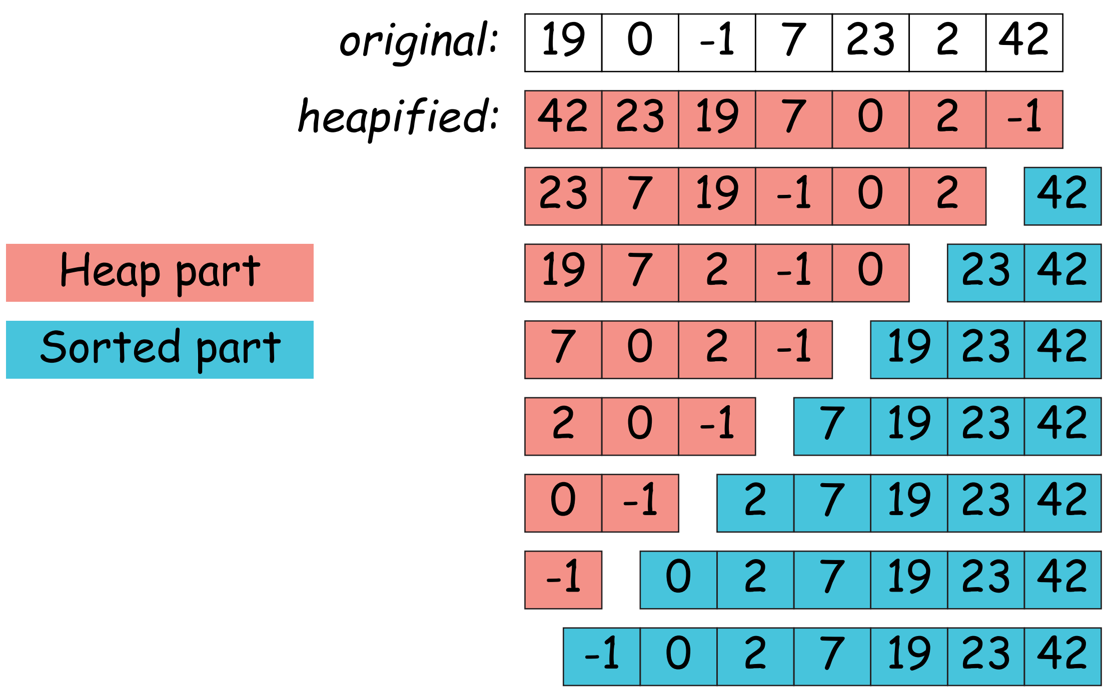

- Clearly this is $\Theta(N^2)$ in the worst case.
- But it is also $\Theta(N^2)$ in the best case, regardless of how sorted the data originally are.

Sorting by Selection: Heapsort

- So straight selection is not such a great idea on a simple list or vector.
- But we've already seen selection in action elsewhere: use heap.
- Gives $O(N \lg N)$ algorithm (N remove-first operations).
- And since we remove items from the end of the heap, we can use that area to accumulate the result:



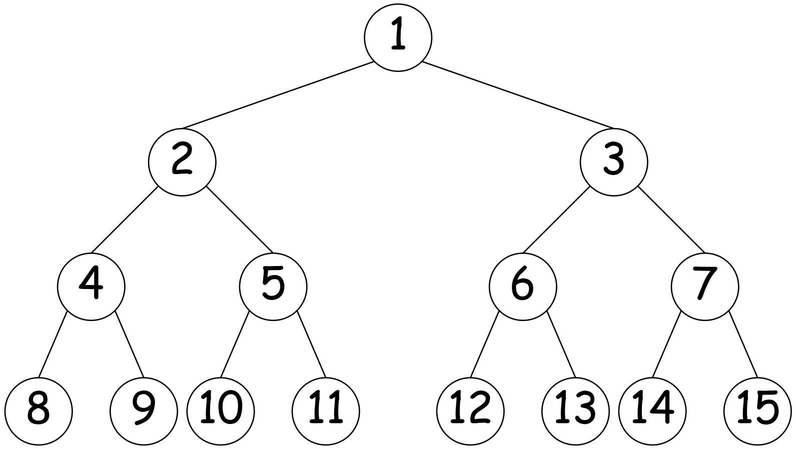
Sorting By Selection: Initial Heapifying

- When covering heaps before, we created them by insertion in an initially empty heap.
- When *given* an array of unheaped data to start with, there is a faster procedure (assume heap indexed from 0):

```
void heapify(int[] arr) {
    int N = arr.length;
    for (int k = N / 2; k >= 0; k -= 1) {
        for (int p = k, c = 0; 2*p + 1 < N; p = c) {
            reheapify downward from p;
        }
    }
}
```

- At each iteration of the p loop, only the element at p might be out of order with respect to its descendants, so reheapifying downward will restore the subtree rooted at p to proper heap ordering.
- Looks like the procedure for re-inserting an element after the top element of the heap is removed, repeated $N/2$ times, *but,...*

Cost of Creating Heap



1 node $\times$ 3 steps down

2 nodes $\times$ 2 steps down

4 nodes $\times$ 1 step down

- In general, worst-case cost for a heap with $h+1$ levels is

$$\begin{aligned}
& 2^0 \cdot h + 2^1 \cdot (h-1) + \ldots + 2^{h-1} \cdot 1 \\
= \; & (2^0 + 2^1 + \ldots + 2^{h-1}) + (2^0 + 2^1 + \ldots + 2^{h-2}) + \ldots + (2^0) \\
= \; & (2^h - 1) + (2^{h-1} - 1) + \ldots + (2^1 - 1) \\
= \; & 2^{h+1} - 1 - h \\
\in \; & \Theta(2^h) = \Theta(N) : \; \textit{linear time}, \text{ not } N \lg N.
\end{aligned}$$

- Alas, since the rest of heapsort still takes $\Theta(N \lg N)$, this does not improve its overall asymptotic cost.