CS61B Lectures #28

Today:

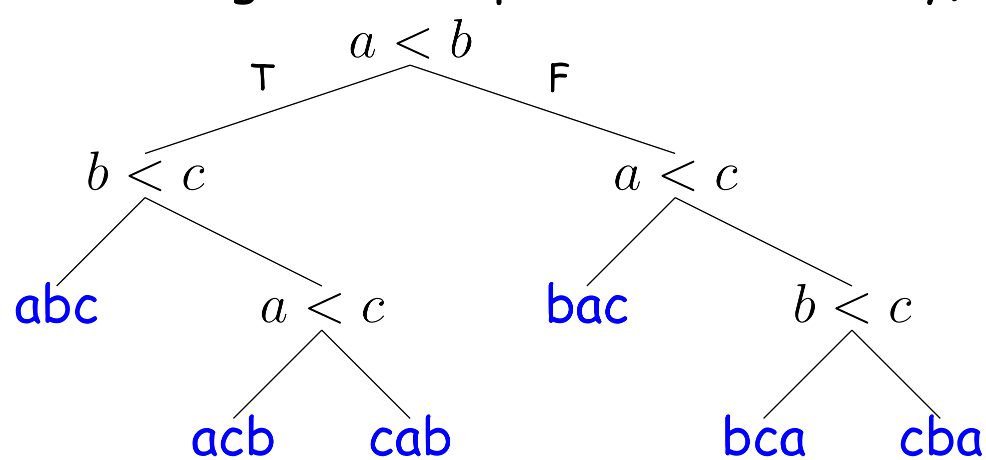
- Lower bounds on sorting by comparison
- Distribution counting, radix sorts

Readings: Today: *DS(IJ)*, Chapter 8; Next topic: Chapter 9.

Better than N lg N?

- We can prove that *if all you can do to keys is compare them,* then sorting must take $\Omega(N \lg N)$.
- Basic idea: there are $N!$ possible ways the input data could be scrambled.
- Therefore, your program must be prepared to do $N!$ different combinations of data-moving operations.
- Therefore, there must be $N!$ possible combinations of outcomes of all the **if**-tests in your program, since those determine what data gets moved where (we're assuming that comparisons are 2-way).

Decision Tree
Height $\propto$ Sorting time



Necessary Choices

- Since each **if**-test goes two ways, the number of possible different outcomes for k **if**-tests is 2^k.
- Thus, there must be enough tests so that $2^k \geq N!$, which means $k \geq \lg N!$.
- Using Stirling's approximation,

$$\begin{aligned}
N! &\in \sqrt{2\pi N}\left(\frac{N}{e}\right)^N\left(1+\Theta\left(\frac{1}{N}\right)\right), \\
\lg(N!) &\in \frac{1}{2}(\lg 2\pi + \lg N) + N\lg N - N\lg e + \lg\left(1+\Theta\left(\frac{1}{N}\right)\right) \\
&= \Theta(N\lg N)
\end{aligned}$$

- This tells us that k, the worst-case number of tests needed to sort N items by comparison sorting, is in $\Omega(N\lg N)$: there must be cases where we need (some multiple of) $N\lg N$ comparisons to sort N things.

Beyond Comparison: Distribution

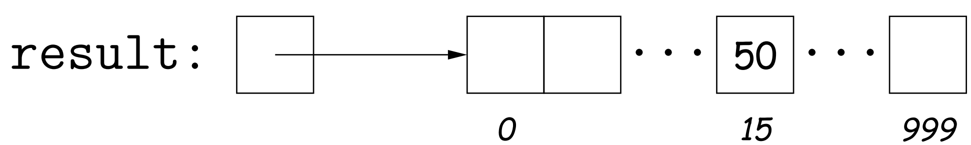
- But suppose we can do more than compare keys?
- For example, how can we sort a set of N different integer keys whose values range from 0 to kN, for some small constant k?
- One technique is *distribution sorting*:
  - Put the integers into N buckets; integer p goes to bucket $\lfloor p/k \rfloor$.
  - There are at most k keys per bucket, so concatenate and use insertion sort, which will now be fast.
- E.g., $k = 2, N = 10:$

```
Start:
  14  3   10  13  4   2   19  17  0   9
In buckets:
  | 0 | 3  2 |  4 |   |  9  | 10 | 13 | 14 |  17 |  19  |
```

- Now insertion sort is fast. Putting the data in buckets takes time $\Theta(N)$, and insertion sort takes $\Theta(kN)$. When k is fixed (constant), we have sorting in time $\Theta(N)$.

Distribution Counting

- Another technique: *count* the number of items < 1, < 2, etc.
- If $M_p =$ #items with value $< p$, then in sorted order, the j^{th} item with value p must be item #$M_p + j$.
- Suppose that one has a set of numbers in the range $[0, 1000)$ (possibly with duplicates) and that exactly 15 of them are less than 50, which is also in the set. Then the result of sorting will look like this:



- In other words, the count of numbers $< k$ gives the index of k in the output array.
- If there are N items in the range $0..M-1$, this gives another *linear-time*—$\Theta(M+N)$—algorithm (We include M and N here to allow for both duplicates and for cases where $M \gg N$.)
- [Postscript on notation: the notations $[A, B]$, (A, B), $[A, B)$, and $(A, B]$ above refer to *intervals*. The use of parentheses vs. square brackets reflects the distinction between *open* and *closed* intervals. Thus $x \in [A, B]$ iff $A \leq x \leq B$, while $x \in [A, B)$ iff $A \leq x < B$, etc.]

Distribution Counting Example

- Suppose all items are between 0 and 9 as in this example:

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum*
<0	<1	<2	<3	<4	<5	<6	<7	<8	<9	

0	0	0	1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9
0			3			6			9		11	12	13			16		

- "Counts" line gives # occurrences of each key.
- "Running sum" gives cumulative count of keys $<$ each value...
- ...which tells us where to put each key:
- The first instance of key k goes into slot m, where m is the number of key instances that are $< k$; next k goes into slot $m+1$, etc.

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Output

0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

0	3	6	7	9	11	12	14	16	16
0	1	2	3	4	5	6	7	8	9

Output

													7					
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

1	3	6	7	9	11	12	14	16	16	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0													7						Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

1	3	6	7	10	11	12	14	16	16	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0									4				7						Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

2	3	6	7	10	11	12	14	16	16	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0								4				7						Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

2	3	6	7	10	11	12	14	16	17	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0								4				7			9			Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	4	6	7	10	11	12	14	16	17
0	1	2	3	4	5	6	7	8	9

Output

0	0		1						4				7			9		
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

0	1	2	3	4	5	6	7	8	9
3	3	1	2	2	1	1	3	0	3

Running sum of Counts

0	1	2	3	4	5	6	7	8	9
0	3	6	7	9	11	12	13	16	16

Next positions

0	1	2	3	4	5	6	7	8	9
2	4	6	7	10	11	12	14	16	18

Output

0			3			6			9			12			15			18
0	0		1						4				7			9	9	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	5	6	7	10	11	12	14	16	18
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1					4				7			9	9	
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

0	1	2	3	4	5	6	7	8	9
3	3	1	2	2	1	1	3	0	3

Running sum of Counts

0	1	2	3	4	5	6	7	8	9
0	3	6	7	9	11	12	13	16	16

Next positions

0	1	2	3	4	5	6	7	8	9
2	5	6	7	10	11	12	14	16	19

Output

0			3			6			9			12			15			18
0	0		1	1					4				7			9	9	9

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	5	6	7	10	12	12	14	16	19
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1					4		5		7			9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

2	5	6	8	10	12	12	14	16	19	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0		1	1			3		4		5		7			9	9	9	Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

2	5	6	8	10	12	12	15	16	19	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0		1	1			3		4		5		7	7		9	9	9	Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	5	6	9	10	12	12	15	16	19
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1			3	3	4		5		7	7		9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	6	6	9	10	12	12	15	16	19
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1	1		3	3	4		5		7	7		9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

2	6	6	9	10	12	13	15	16	19	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0		1	1	1		3	3	4		5	6	7	7		9	9	9	Output
0			3			6			9			12			15			18	

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	6	6	9	10	12	13	16	16	19
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1	1		3	3	4		5	6	7	7	7	9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

3	3	1	2	2	1	1	3	0	3
0	1	2	3	4	5	6	7	8	9

Running sum of Counts

0	3	6	7	9	11	12	13	16	16
0	1	2	3	4	5	6	7	8	9

Next positions

2	6	6	9	11	12	13	16	16	19
0	1	2	3	4	5	6	7	8	9

Output

0	0		1	1	1		3	3	4	4	5	6	7	7	7	9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

Counts

0	1	2	3	4	5	6	7	8	9
3	3	1	2	2	1	1	3	0	3

Running sum of Counts

0	1	2	3	4	5	6	7	8	9
0	3	6	7	9	11	12	13	16	16

Next positions

0	1	2	3	4	5	6	7	8	9
2	6	7	9	11	12	13	16	16	19

Output

0	0		1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9
0			3			6			9			12			15			18

Distribution Counting Example (II)

7	0	4	0	9	1	9	1	9	5	3	7	3	1	6	7	4	2	0

3	3	1	2	2	1	1	3	0	3	*Counts*
0	1	2	3	4	5	6	7	8	9	

0	3	6	7	9	11	12	13	16	16	*Running sum of Counts*
0	1	2	3	4	5	6	7	8	9	

3	6	7	9	11	12	13	16	16	19	*Next positions*
0	1	2	3	4	5	6	7	8	9	

0	0	0	1	1	1	2	3	3	4	4	5	6	7	7	7	9	9	9	Output
0			3			6			9			12			15			18	

Radix Sort

Idea: Sort keys *one character at a time.*

- We can use distribution counting for each digit.
- Can work either right to left (LSD radix sort) or left to right (MSD radix sort)
- LSD radix sort is venerable: used for punched cards. Example:

Initial: set, cat, cad, con, bat, can, be, let, bet

Pass 1 (by char #2)

			bet
			let
			bat
		can	cat
be	cad	con	set
'␣'	'd'	'n'	't'

be, cad, con, can, set, cat, bat, let, bet

Pass 2 (by char #1)

bat	bet	
cat	let	
can	set	
cad	be	con
'a'	'e'	'o'

cad, can, cat, bat, be, set, let, bet, con

Pass 3 (by char #0)

	con		
bet	cat		
be	can		
bat	cad	let	set
'b'	'c'	'l'	's'

bat, be, bet, cad, can, cat, con, let, set

MSD Radix Sort

- A bit more complicated: must keep lists from each step separate,
- But, we can stop processing 1-element lists

A	posn
$\star$ set, cat, cad, con, bat, can, be, let, bet	0
$\star$ bat, be, bet / cat, cad, con, can / let / set	1
bat / $\star$ be, bet / cat, cad, con, can / let / set	2
bat / be / bet / $\star$ cat, cad, con, can / let / set	1
bat / be / bet / $\star$ cat, cad, can / con / let / set	2
bat / be / bet / cad / can / cat / con / let / set	

- Here, slashes divide partially sorted sublists, which will never be moved into the spaces occupied by other sublists.
- Asterisks mark a sublist to be sorted on some character position.

Performance of Radix Sort

- Radix sort takes $\Theta(B)$ time where B is *the total size of the key data (total characters)*.
- Up until now, we have measured sort times as functions of #records.
- How to compare these two different measures?
- To have N different records, one must have keys at least $\Theta(\lg N)$ long [why?]
- Furthermore, comparisons actually take time $\Theta(K)$ where K is the size of key in worst case [why?]
- So $N \lg N$ comparisons really means $N(\lg N)^2$ operations.
- While radix sort would take $B = N \lg N$ time for N records with minimal-length ($\lg N$) keys.
- On the other hand, we must work to get good constant factors with radix sort.

And Don't Forget Search Trees

Idea: A search tree is in sorted order, when read in inorder.

- Need *balance* to really use for sorting [next topic].
- Given balance, same performance as heapsort: N insertions in time $\lg N$ each, plus $\Theta(N)$ to traverse, gives

$$\Theta(N + N \lg N) = \Theta(N \lg N)$$

Summary

- Insertion sort: $\Theta(Nk)$ comparisons and moves, where k is maximum amount data is displaced from final position.
  - Good for small datasets or almost ordered data sets.
- Quicksort: $\Theta(N \lg N)$ with good constant factor if data is not pathological. Worst case $O(N^2)$.
- Merge sort: $\Theta(N \lg N)$ guaranteed. Good for external sorting.
- Heapsort, treesort with guaranteed balance: $\Theta(N \lg N)$ guaranteed.
- Radix sort, distribution sort: $\Theta(B)$ (number of bytes). Also good for external sorting.