CS61B Lecture #33

Today's Readings: Graph Structures: *DSIJ*, Chapter 12

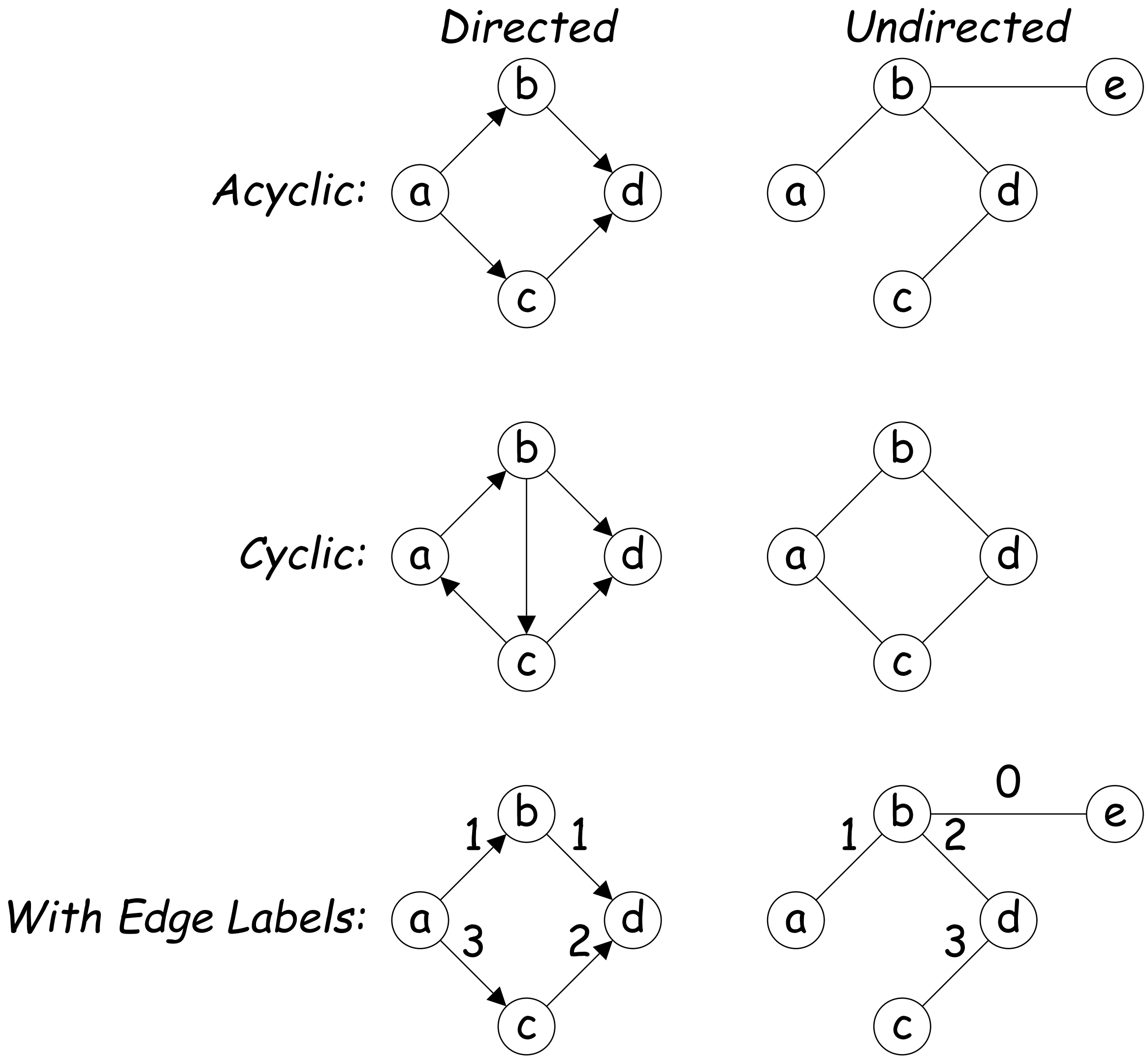
Why Graphs?

- For expressing non-hierarchically related items
- Examples:
  - Networks: pipelines, roads, assignment problems
  - Representing processes: flow charts, Markov models
  - Representing partial orderings: PERT charts, makefiles
  - As we've seen, in representing connected structures as used in Git.

Some Terminology

- A *graph* consists of
  - A set of *nodes* (aka *vertices*)
  - A set of *edges:* pairs of nodes.
  - Nodes with an edge between are *adjacent*.
  - Depending on problem, nodes or edges may have *labels* (or *weights*)
- Typically call the node set $V = \{v_0, \ldots\}$, and the edge set E.
- If the edges have an order (first, second), they are *directed edges,* and we have a *directed graph (digraph);* otherwise an *undirected graph*.
- Edges are *incident* to their nodes.
- Directed edges *exit* one node and *enter* the next.
- A *cycle* is a *path*—a sequence of edges—without repeated edges leading from a node back to itself (following arrows if directed).
- A graph is *cyclic* if it has a cycle; else *acyclic*. Abbreviation: Directed Acyclic Graph—*DAG*.

Some Pictures



Directed

Undirected

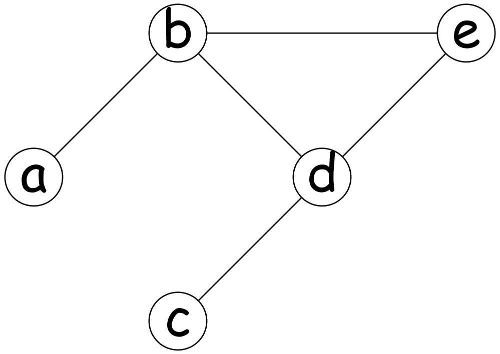
Acyclic:

Cyclic:

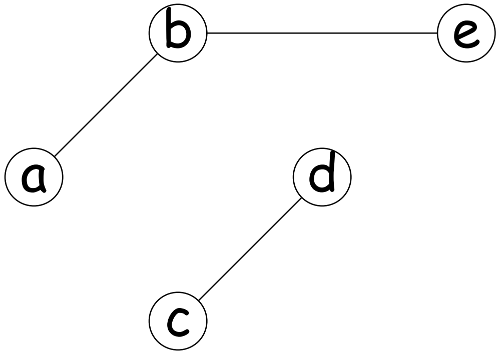
With Edge Labels:

Connectivity (Undirected)

- A *path* is variously defined as a sequence of vertices $v_0, v_1, \ldots, v_n$ where there is an edge from each v_i to v_{i+1}, or as the sequence of edges between them: $(v_0, v_1), (v_1, v_2), \ldots$.
- An undirected graph is *connected* if there is a *path* between every pair of nodes in the graph:



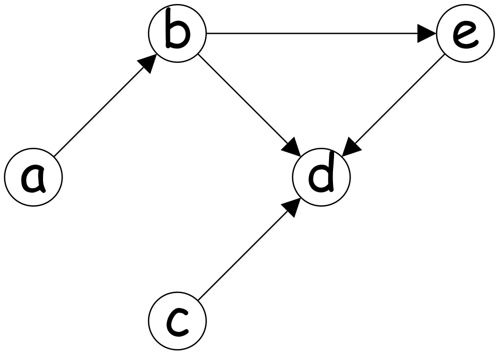
Connected



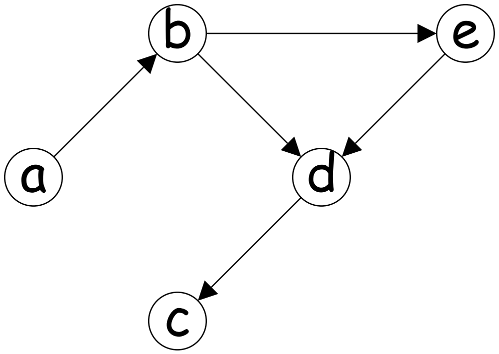
Unconnected

Connectivity (Directed)

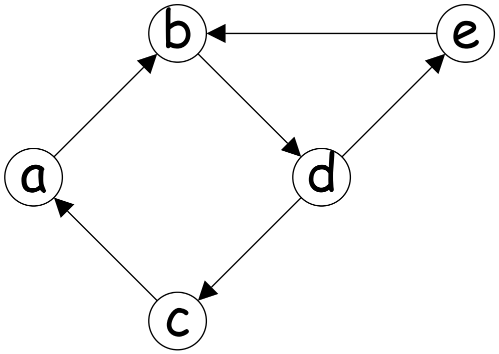
- For directed graphs, it's more complicated:
  - *Weakly connected:* connected if direction is removed.
  - *Unilaterally connected* (or *semiconnected*): there is a path in one direction or the other between each node pair;
  - *Strongly connected*: there are paths in both directions for each node pair.



Weakly Connected



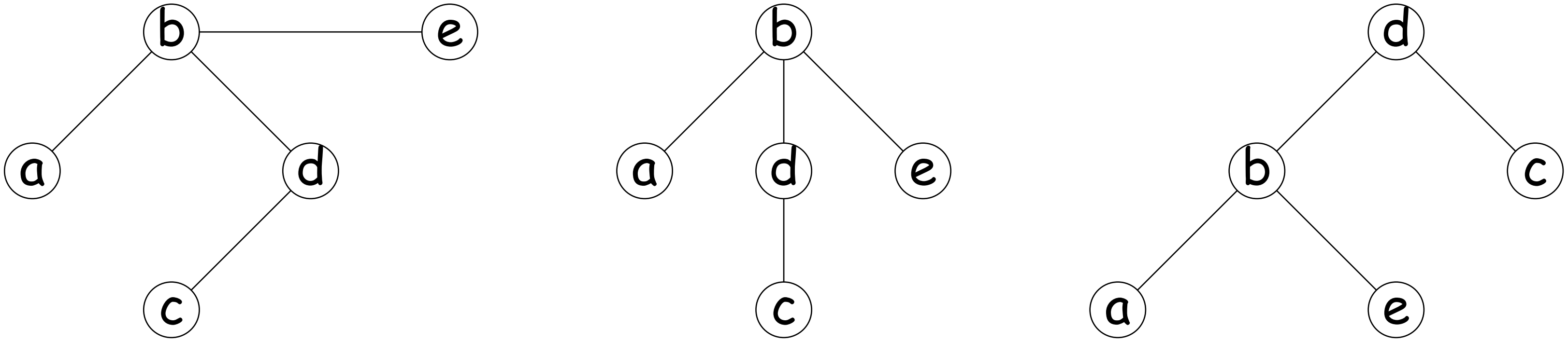
Unilaterally Connected



Strongly Connected

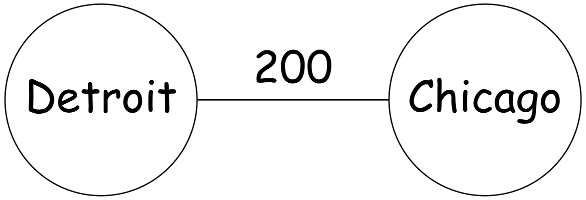
Trees and Graphs

- A DAG is a (rooted) tree iff connected, and every node but the root has exactly one parent.
- A connected, acyclic, undirected graph is also called a *free tree.* Free: we're free to pick the root; e.g., all the following are the same graph:

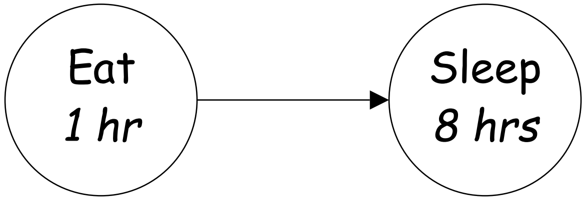


Examples of Use

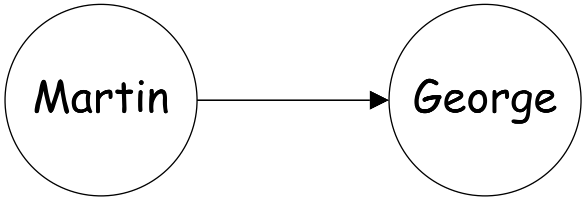
- Edge = Connecting road, with length.



- Edge = Must be completed before; Node label = time to complete.

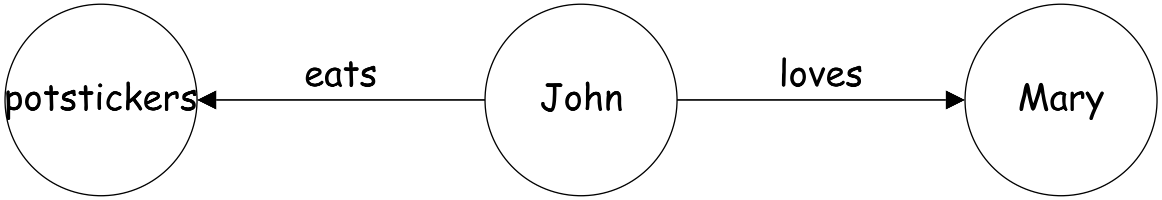


- Edge = Begat

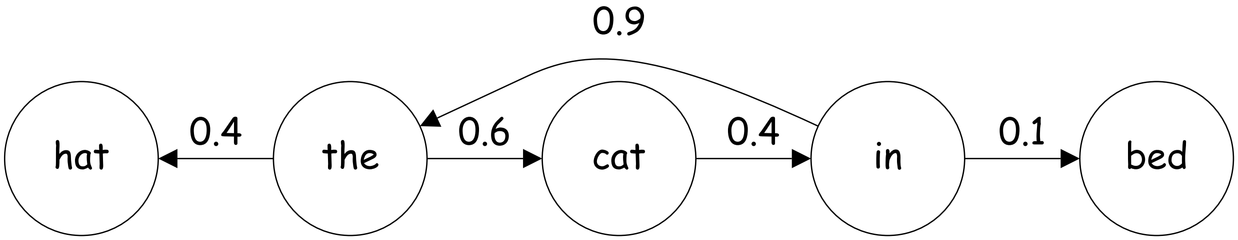


More Examples

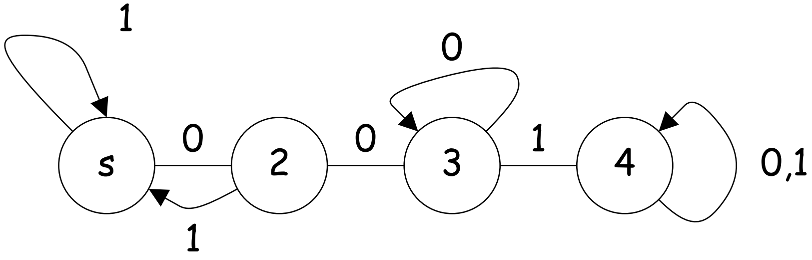
- Edge = some relationship



- Edge = next state might be (with probability)

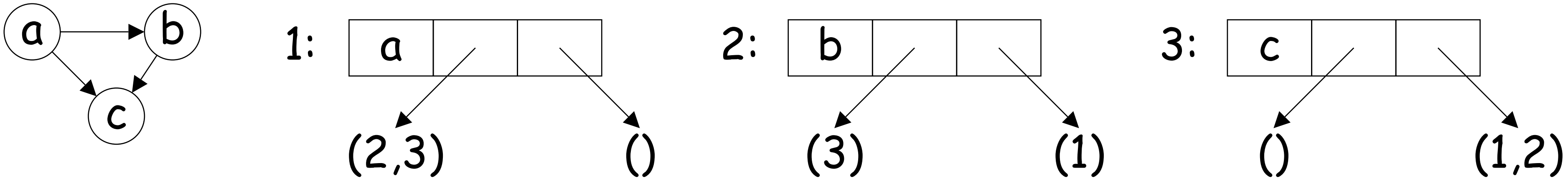


- Edge = next state in state machine, label is triggering input. (Start at s. Being in state 4 means "there is a substring '001' somewhere in the input".)



Representation

- Often useful to number the nodes, and use the numbers in edges.
- *Edge list representation:* each node contains some kind of list (e.g., linked list or array) of its successors (and possibly predecessors).



- *Edge sets:* Collection of all edges. For graph above:

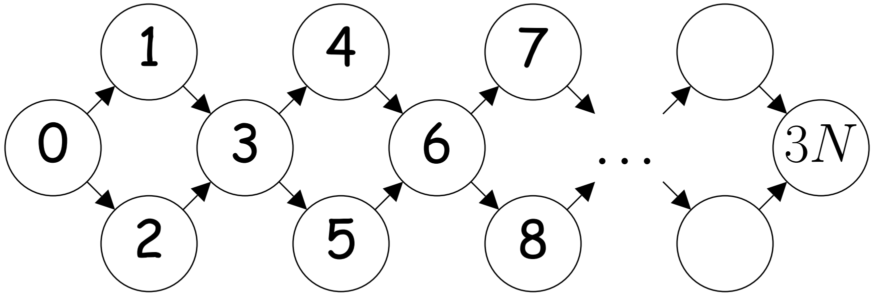
$$\{(1,2),(1,3),(2,3)\}$$

- *Adjacency matrix:* Represent connection with matrix entry:

$$\begin{array}{cc} & \mathit{1} \quad \mathit{2} \quad \mathit{3} \\ \mathit{1} & \\ \mathit{2} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \mathit{3} & \end{array}$$

Traversing a Graph

- Many algorithms on graphs depend on traversing all or some nodes.
- Can't quite use recursion because of cycles.
- Even in acyclic graphs, can get combinatorial explosions:



  Treat 0 as the root and do recursive traversal down the two edges out of each node: $\Theta(2^N)$ operations!

- So typically try to visit each node constant # of times (e.g., once).

Recursive Depth-First Traversal of a Graph

- Can fix looping and combinatorial problems using the “bread-crumb” method used in earlier lectures for a maze.
- That is, *mark* nodes as we traverse them and don’t traverse previously marked nodes.
- Makes sense to talk about *preorder* and *postorder*, as for trees.

```
void preorderTraverse(Graph G, Node v)
{
   if (v is unmarked) {
     mark(v);
     visit v;
     for (Edge(v, w) ∈ G)
       traverse(G, w);
   }
}
```

```
void postorderTraverse(Graph G, Node v)
{
   if (v is unmarked) {
     mark(v);
     for (Edge(v, w) ∈ G)
       traverse(G, w);
     visit v;
   }
}
```

Recursive Depth-First Traversal of a Graph (II)

- We are often interested in traversing *all* nodes of a graph, not just those reachable from one node.
- So we can repeat the procedure as long as there are unmarked nodes.

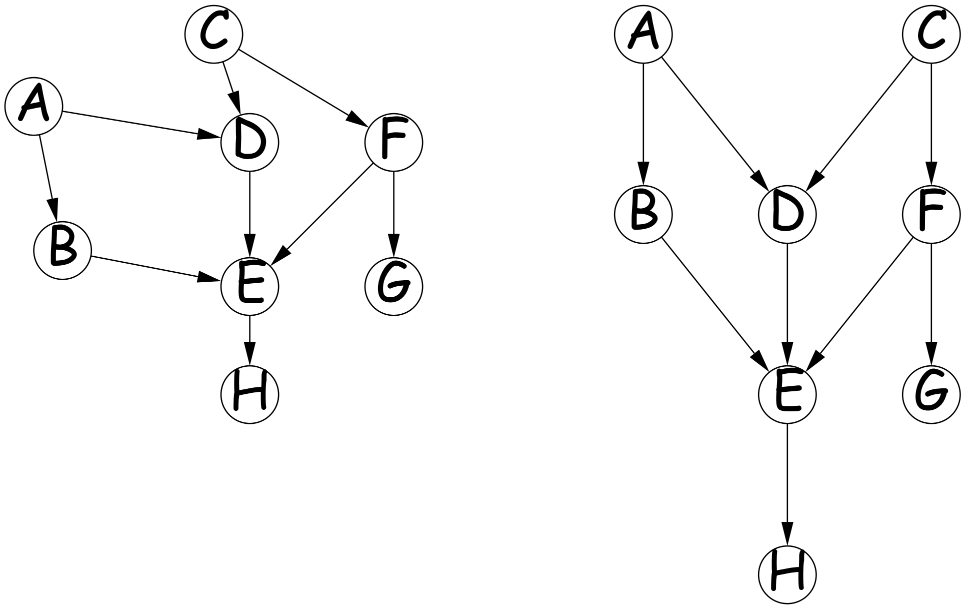
```
void preorderTraverse(Graph G) {
    clear all marks;
    for (v ∈ nodes of G) {
        preorderTraverse(G, v);
    }
}

void postorderTraverse(Graph G) {
    clear all marks;
    for (v ∈ nodes of G) {
        postorderTraverse(G, v);
    }
}
```

Topological Sorting

Problem: Given a DAG, find a linear order of nodes consistent with the edges.

- That is, order the nodes $v_0,\ v_1,\ \ldots$ such that v_k is never reachable from $v_{k'}$ if $k' > k$.
- Gmake does this. Also PERT charts.

Graph (two views)

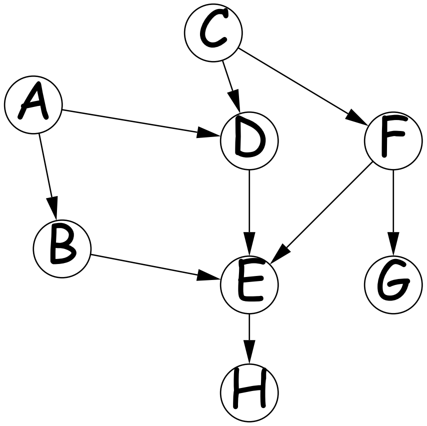


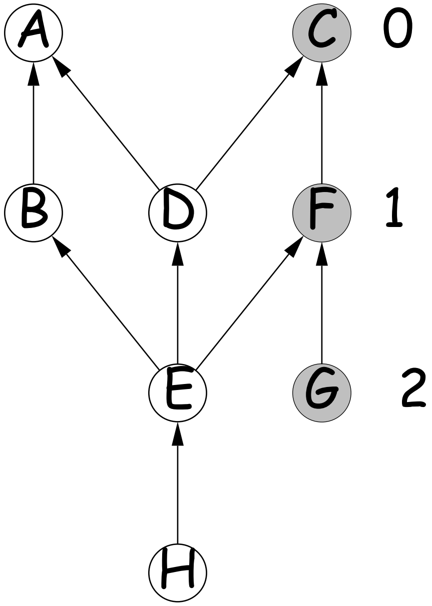
Possible Orderings

A	C	C
C	A	F
B	F	G
D	D	A
F	B	B
E	G	D
G	E	E
H	H	H

Sorting and Depth First Search

- **Observation**: Suppose we *reverse the links* on our graph.
- If we do a recursive DFS on the reverse graph, starting from node H, for example, we will find all nodes that must come *before* H.
- When the search reaches a node in the reversed graph and there are no successors, we know that it is safe to put that node first.
- In general, a *postorder* traversal of the *reversed* graph visits nodes only after all predecessors have been visited.





Numbers show post-order traversal order starting from G: everything that must come before G.

General Graph Traversal Algorithm

```
COLLECTION_OF_VERTICES fringe;

fringe = INITIAL_COLLECTION;
while (!fringe.isEmpty()) {
  Vertex v = fringe.REMOVE_HIGHEST_PRIORITY_ITEM();

  if (!MARKED(v)) {
    MARK(v);
    VISIT(v);
    For each edge(v,w) {
      if (NEEDS_PROCESSING(w))
        Add w to fringe;
    }
  }
}
```

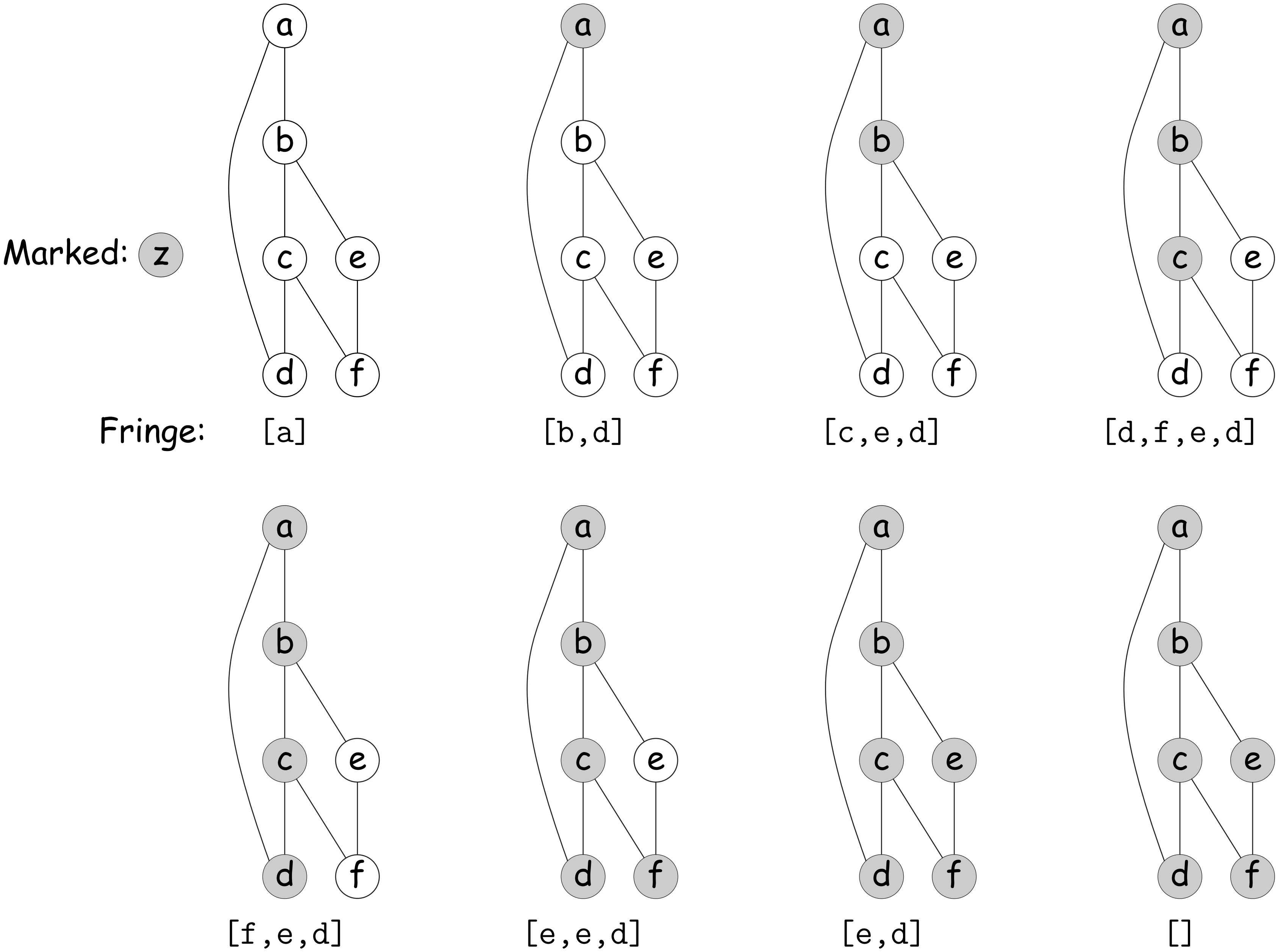
Replace *COLLECTION_OF_VERTICES*, *INITIAL_COLLECTION*, etc. with various types, expressions, or methods to different graph algorithms.

Example: Depth-First Traversal

Problem: Visit every node reachable from v once, visiting nodes further from start first.

```
// Red sections are specializations of general algorithm
Stack<Vertex> fringe;

fringe = stack containing {v};
while (!fringe.isEmpty()) {
  Vertex v = fringe.pop();

  if (!marked(v)) {
    mark(v);
    VISIT(v);
    For each edge(v,w) {
      if (!marked(w))
        fringe.push(w);
    }
  }
}
```

Depth-First Traversal Illustrated



Marked: z

Fringe: [a] [b,d] [c,e,d] [d,f,e,d]

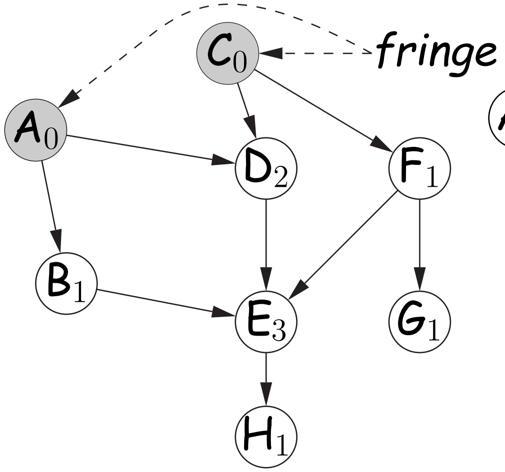
[f,e,d] [e,e,d] [e,d] []

Topological Sort Revisited

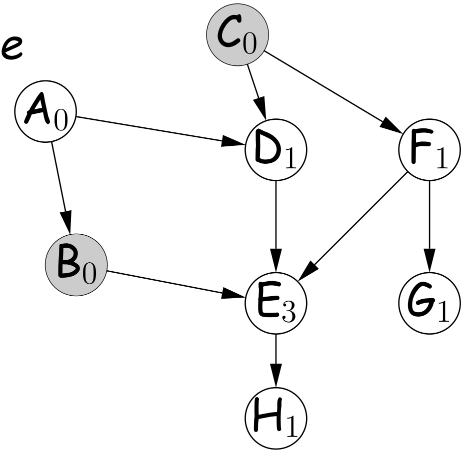
- Another approach to topological sorting uses the general traveral algorithm.
- We keep a sequence of output nodes, initialized to empty.
- For each node, maintain a count of the number of immediate predecessor nodes (the number of nodes with edges going to that node) that have not yet been output.
- At each step, output a node with no remaining predecessors, and update the predecessor counts.

```
Set<Vertex> fringe;
Map<Vertex, Integer> predCount;
fringe = empty set of vertices;
predCount = map from each vertex to number of predecessors;
while (!fringe.isEmpty()) {
  Vertex v = fringe.removeSomeElement();  /* Not a real operation! */
  Add v to the output;
  for each edge(v,w) {
      predCount.put(w, predCount.get(w) - 1);
      if (predCount.get(w) == 0) fringe.add(w);
  }
}
```

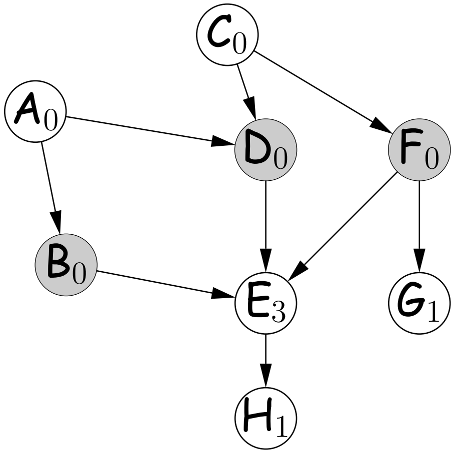
Alternative Topological Sort in Action



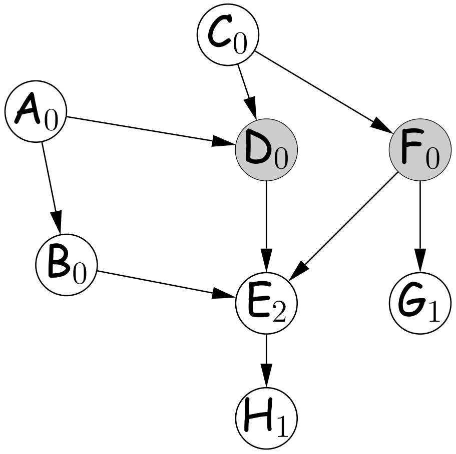
Output: `[]`



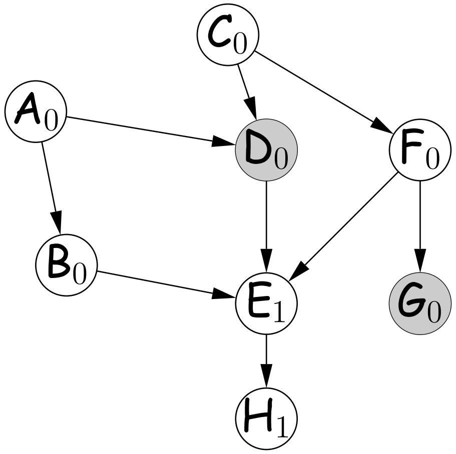
`[A]`



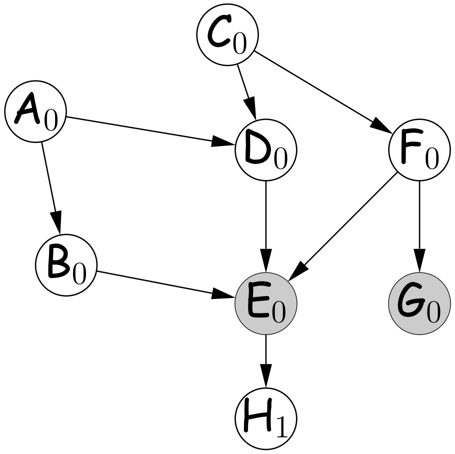
`[A,C]`



`[A,C,B]`

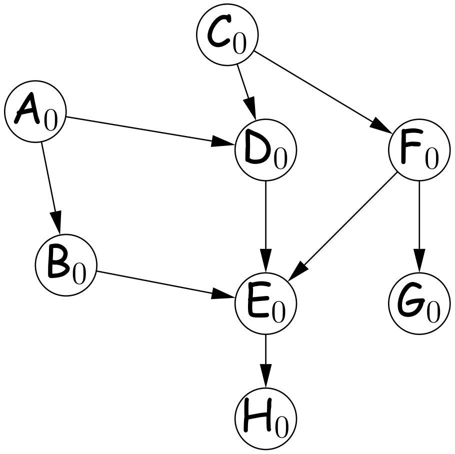


`[A,C,B,F]`



`[A,C,B,F,D]`

...



`[A,C,B,F,D,E,G,H]`

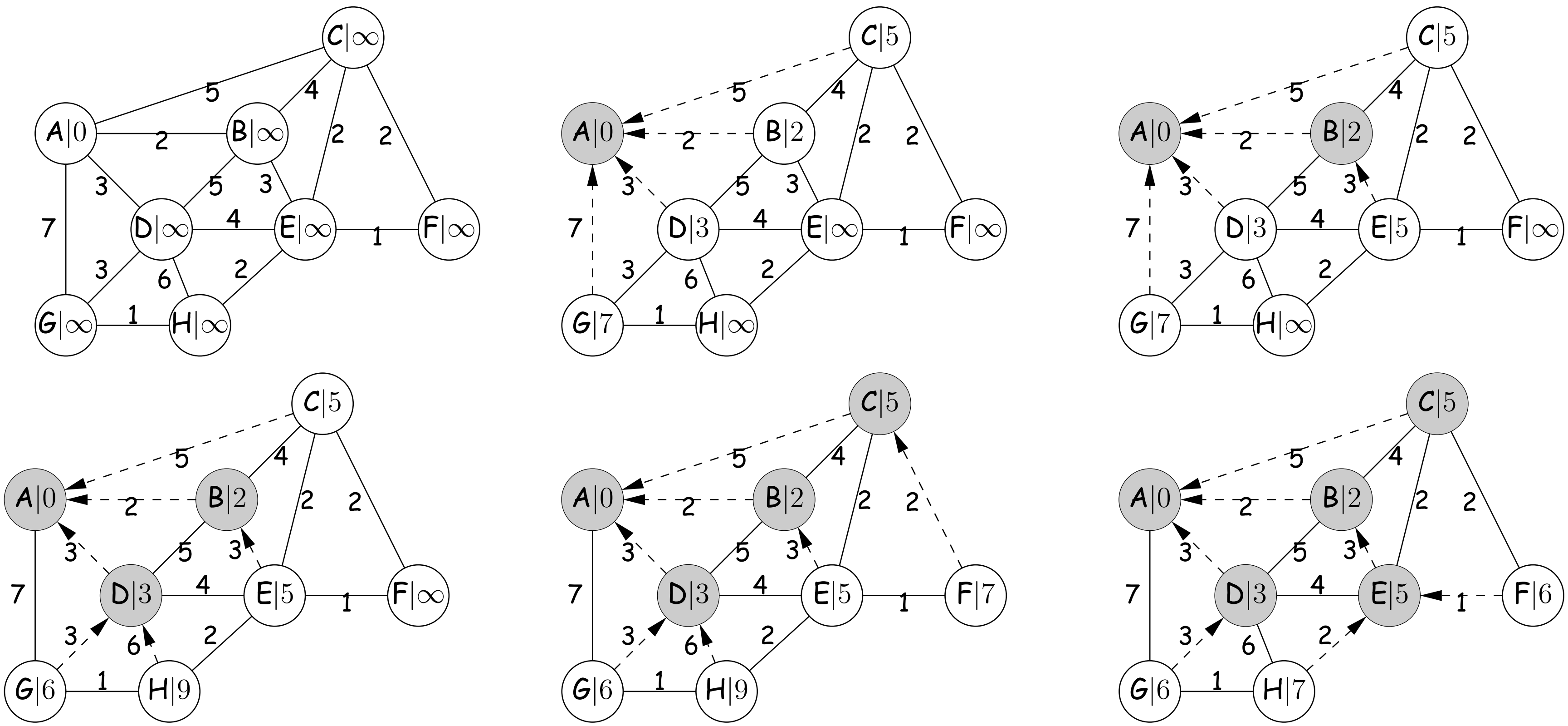
Shortest Paths: Dijkstra's Algorithm

Problem: Given a graph (directed or undirected) with non-negative edge weights, compute shortest paths from given source node, s, to all nodes.

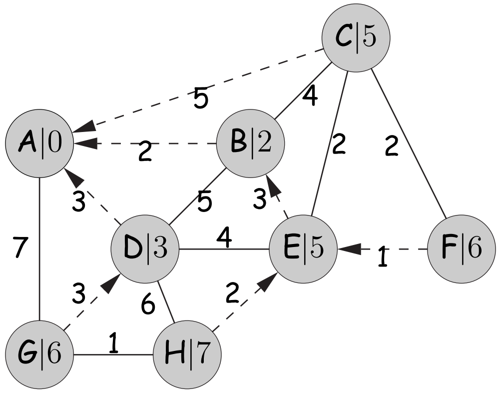
- "Shortest" = sum of weights along path is smallest.
- For each node, keep estimated distance from s, ...
- ...and of preceding node in shortest path from s.

```
PriorityQueue<Vertex> fringe;
For each node v { v.dist() = ∞; v.back() = null; }
s.dist() = 0;
fringe = priority queue ordered by smallest .dist();
add all vertices to fringe;
while (!fringe.isEmpty()) {
  Vertex v = fringe.removeFirst();

  For each edge(v,w) {
    if (v.dist() + weight(v,w) < w.dist())
      { w.dist() = v.dist() + weight(v,w); w.back() = v; }
  }
}
```

Example



Final result:



- - -► Shortest-path tree

● X|d processed node at distance d

○ Y|d node in fringe at distance d