CS61C – Machine Structures

Lecture 2 – Number Representation

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From Lecture 1

- Read class info document.
- Get class account form and login by 5pm today.
- Look at class website often!
- Posted / handouts:
  - Reading for this week: P&H Ch1 & 3.1-2
  - Reading for Monday K&R Ch1-4
  - Reading for W, F: K&R Ch5&6
  - HW1 due Wednesday

Decimal Numbers: Base 10

Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Example:

3271 =

$$(3 \times 10^3) + (2 \times 10^2) + (7 \times 10^1) + (1 \times 10^0)$$

Numbers: positional notation

- Number Base B ⇒ B symbols per digit:
  - Base 10 (Decimal): 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
    Base 2 (Binary): 0, 1
- Number representation:
  - $d_{31}d_{30}$ ... $d_2d_1d_0$ is a 32 digit number
  - value = $d_{31} \times B^{31} + d_{30} \times B^{30} + ... + d_2 \times B^2 + d_1 \times B^1 + d_0 \times B^0$
- Binary: 0,1 (In binary digits called "bits")
  - $1011010 = 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 0 \times 1 = 64 + 16 + 8 + 2 = 90$
  - Notice that 7 digit binary number turns into a 2 digit decimal number
  - A base that converts to binary easily?

Hexadecimal Numbers: Base 16

- Hexadecimal:
  0,1,2,3,4,5,6,7,8,9, A, B, C, D, E, F
  - Normal digits + 6 more from the alphabet
- Conversion: Binary⇔Hex
  - 1 hex digit represents 16 decimal values
  - 4 binary digits represent 16 decimal values
  - ⇒ 1 hex digit replaces 4 binary digits
- Example:
  - 1010 1100 0101 (binary) = ? (hex)

Decimal vs. Hexadecimal vs. Binary

Examples:

1010 1100 0101 (binary)
= AC5 (hex)

10111 (binary)
= 0001 0111 (binary)
= 17 (hex)

3F9(hex)
= 11 1111 1001 (binary)

How do we convert between hex and Decimal?

00	0	0000
01	1	0001
02	2	0010
03	3	0011
04	4	0100
05	5	0101
06	6	0110
07	7	0111
08	8	1000
09	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

What to do with representations of numbers?

- Just what we do with numbers!
  - Add them
  - Subtract them
  - Multiply them
  - Divide them
  - Compare them
- Example: 10 + 7 = 17

```
    1  1
    1  0  1  0
+   0  1  1  1
-------------------
1   0  0  0  1
```

  - so simple to add in binary that we can build circuits to do it
  - subtraction also just as you would in decimal

Comparison

How do you tell if X > Y ?

Which base do we use?

- Decimal: great for humans, especially when doing arithmetic
- Hex: if human looking at long strings of binary numbers, its much easier to convert to hex and look 4 bits/symbol
  - Terrible for arithmetic on paper
- Binary: what computers use; you will learn how computers do +,-,*,/
  - To a computer, numbers always binary
  - Regardless of how number is written:

$$32_{10} == 0x20 == 100000_2$$

  - Use subscripts "ten", "hex", "two" in book, slides when might be confusing

Limits of Computer Numbers

- Bits can represent anything!
- Characters?
  - 26 letters ⇒ 5 bits (2^5 = 32)
  - upper/lower case + punctuation ⇒ 7 bits (in 8) ("ASCII")
  - standard code to cover all the world's languages ⇒ 16 bits ("unicode")
- Logical values?
  - 0 ⇒ False, 1 ⇒ True
- colors ? Ex: *Red (00)* *Green (01)* *Blue (11)*
- locations / addresses? commands?
- but N bits ⇒ only 2^N things

How to Represent Negative Numbers?

- So far, unsigned numbers
- Obvious solution: define leftmost bit to be sign!
  - 0 ⇒ +, 1 ⇒ -
  - Rest of bits can be numerical value of number
- Representation called sign and magnitude
- MIPS uses 32-bit integers. $+1_{ten}$ would be:

0000 0000 0000 0000 0000 0000 0000 0001

- And - 1_{ten} in sign and magnitude would be:

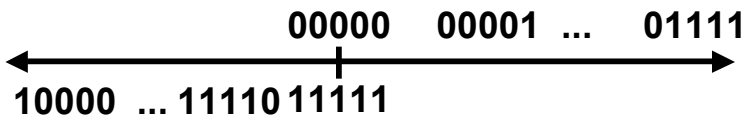
1000 0000 0000 0000 0000 0000 0000 0001

Shortcomings of sign and magnitude?

- Arithmetic circuit complicated
  - Special steps depending whether signs are the same or not
- Also, Two zeros
  - 0x00000000 = $+0_{ten}$
  - 0x80000000 = -0_{ten}
  - What would 2 0s mean for programming?
- Therefore sign and magnitude abandoned

Another try: complement the bits

- Example: $7_{10} = 00111_2$ $-7_{10} = 11000_2$
- Called One's Complement
- Note: positive numbers have leading 0s, negative numbers have leadings 1s.



- What is -00000 ? Answer: 11111
- How many positive numbers in N bits?
- How many negative ones?

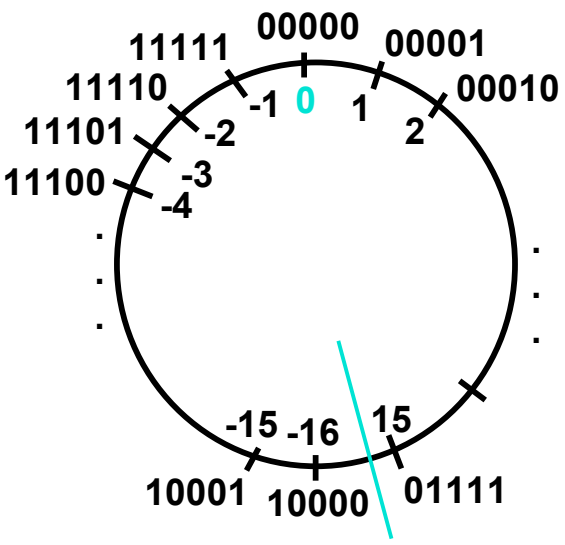
Shortcomings of One's complement?

- Arithmetic still a somewhat complicated.
- Still two zeros
  - `0x00000000` = $+0_{ten}$
  - `0xFFFFFFFF` = -0_{ten}
- Although used for awhile on some computer products, one's complement was eventually abandoned because another solution was better.

Standard Negative Number Representation

- What is result for unsigned numbers if tried to subtract large number from a small one?
  - Would try to borrow from string of leading 0s, so result would have a string of leading 1s
    - 3 - 4 ⇒ 00…0011 - 00…0100 = 11…1111
  - With no obvious better alternative, pick representation that made the hardware simple
  - As with sign and magnitude, leading 0s ⇒ positive, leading 1s ⇒ negative
    - 000000...xxx is >=0, 111111...xxx is < 0
    - except 1…1111 is -1, not -0 (as in sign & mag.)
- This representation is Two's Complement

2's Complement Number "line": N = 5



- 2^{N-1} non-negatives
- 2^{N-1} negatives
- one zero
- how many positives?

Two's Complement for N=32

0000 ... 0000 0000 0000 0000$_{two}$ =	0$_{ten}$
0000 ... 0000 0000 0000 0001$_{two}$ =	1$_{ten}$
0000 ... 0000 0000 0000 0010$_{two}$ =	2$_{ten}$
. . .	
0111 ... 1111 1111 1111 1101$_{two}$ =	2,147,483,645$_{ten}$
0111 ... 1111 1111 1111 1110$_{two}$ =	2,147,483,646$_{ten}$
0111 ... 1111 1111 1111 1111$_{two}$ =	2,147,483,647$_{ten}$
1000 ... 0000 0000 0000 0000$_{two}$ =	–2,147,483,648$_{ten}$
1000 ... 0000 0000 0000 0001$_{two}$ =	–2,147,483,647$_{ten}$
1000 ... 0000 0000 0000 0010$_{two}$ =	–2,147,483,646$_{ten}$
. . .	
1111 ... 1111 1111 1111 1101$_{two}$ =	–3$_{ten}$
1111 ... 1111 1111 1111 1110$_{two}$ =	–2$_{ten}$
1111 ... 1111 1111 1111 1111$_{two}$ =	–1$_{ten}$

- One zero; 1st bit called sign bit
- 1 "extra" negative:no positive 2,147,483,648$_{ten}$

Two's Complement Formula

- Can represent positive and negative numbers in terms of the bit value times a power of 2:

$$d_{31} \times (-2^{31}) + d_{30} \times 2^{30} + \ldots + d_2 \times 2^2 + d_1 \times 2^1 + d_0 \times 2^0$$

- Example: 1111 1100$_{two}$

$$= 1\times(-2^7) + 1\times2^6 + 1\times2^5 + 1\times2^4 + 1\times2^3 + 1\times2^2 + 0\times2^1 + 0\times2^0$$

$$= -2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 0 + 0$$

$$= -128 + 64 + 32 + 16 + 8 + 4$$

$$= -128 + 124$$

$$= -4_{ten}$$

Two's Complement shortcut: Negation

- Change every 0 to 1 and 1 to 0 (invert or complement), then add 1 to the result
- Proof: Sum of number and its (one's) complement must be $111...111_{two}$
  - However, $111...111_{two} = -1_{ten}$
  - Let x' ⇒ one's complement representation of x
  - Then x + x' = -1 ⇒ x + x' + 1 = 0 ⇒ x' + 1 = -x
- Example: -4 to +4 to -4

x : 1111 1111 1111 1111 1111 1111 1111 1100_{two}
x': 0000 0000 0000 0000 0000 0000 0000 0011_{two}
+1: 0000 0000 0000 0000 0000 0000 0000 0100_{two}
()': 1111 1111 1111 1111 1111 1111 1111 1011_{two}
+1: 1111 1111 1111 1111 1111 1111 1111 1100_{two}

Two's comp. shortcut: Sign extension

- Convert 2's complement number rep. using n bits to more than n bits
- Simply replicate the most significant bit (sign bit) of smaller to fill new bits
  - 2's comp. positive number has infinite 0s
  - 2's comp. negative number has infinite 1s
  - Binary representation hides leading bits; sign extension restores some of them
  - 16-bit -4_{ten} to 32-bit:

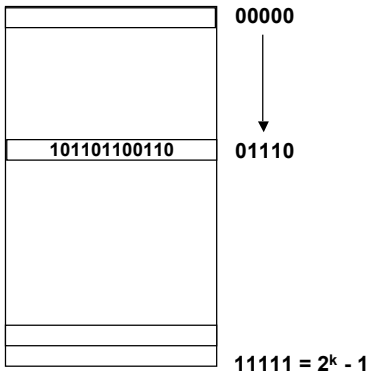
$1111\ 1111\ 1111\ 1100_{two}$

$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1100_{two}$

Signed vs. Unsigned Variables

- Java just declares integers `int`
  - Uses two's complement
- C has declaration `int` also
  - Declares variable as a signed integer
  - Uses two's complement
- Also, C declaration `unsigned int`
  - Declares a unsigned integer
  - Treats 32-bit number as unsigned integer, so most significant bit is part of the number, not a sign bit

Numbers represented in memory



- Memory is a place to store bits
- A *word* is a fixed number of bits (eg, 32) at an address
- *Addresses* are naturally represented as unsigned numbers in C

Signed v. Unsigned Comparisons

- X = 1111 1111 1111 1111 1111 1111 1111 1100_{two}
- Y = 0011 1011 1001 1010 1000 1010 0000 0000_{two}

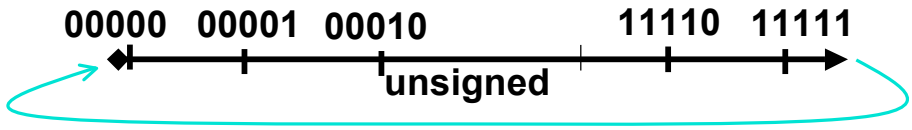
- Is X > Y?

  unsigned: YES

  signed: NO

What if too big?

- Binary bit patterns above are simply representatives of numbers. Strictly speaking they are called "numerals".
- Numbers really have an infinite number of digits
  - with almost all being same (00…0 or 11…1) except for a few of the rightmost digits
  - Just don't normally show leading digits
- If result of add (or -,*,/) cannot be represented by these rightmost HW bits, overflow is said to have occurred.



And in Conclusion...

- We represent "things" in computers as particular bit patterns: N bits ⇒ 2^N
  - numbers, characters, ...
- Decimal for human calculations, binary to understand computers, hexadecimal to understand binary
- 2's complement universal in computing: cannot avoid, so learn
- Overflow: numbers infinite but computers finite, so errors occur