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UC Berkeley CS61C : Machine Structures

Lecture 16
Floating Point II

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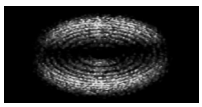
Electron Filmed for the First Time



"Scientists have filmed an electron in motion for the first time, using a new technique that will allow researchers to study the tiny particle's movements directly."

"Extremely short flashes of light are necessary to capture an electron in motion. A technology developed within the last few years can generate short pulses of intense laser light, called attosecond pulses, to get the job done."

http://www.livescience.com/strangenews/080225-electron-movie.html

Review

Exponent tells Significand how much (2^i) to count by (…, 1/4, 1/2, 1, 2, …)

- **Floating Point lets us:**
  - **Represent numbers containing both integer and fractional parts; makes efficient use of available bits.**
  - **Store approximate values for very large and very small #s.**
- **IEEE 754 Floating Point Standard is most widely accepted attempt to standardize interpretation of such numbers (Every desktop or server computer sold since ~1997 follows these conventions)**

Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

$(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$

Double precision identical, except with exponent bias of 1023 (half, quad similar)

"Father" of the Floating point standard

IEEE Standard 754 for Binary Floating-Point Arithmetic.

1989 ACM Turing Award Winner!

Prof. Kahan

www.cs.berkeley.edu/~wkahan/
…/ieee754status/754story.html

Precision and Accuracy

Don't confuse these two terms!

Precision is a count of the number bits in a computer word used to represent a value.

Accuracy is a measure of the difference between the actual value of a number and its computer representation.

High precision permits high accuracy but doesn't guarantee it. It is possible to have high precision but low accuracy.

Example: `float pi = 3.14;`

pi will be represented using all 24 bits of the significant (highly precise), but is only an approximation (not accurate).

Representation for ± ∞

- **In FP, divide by 0 should produce ± ∞, not overflow.**
- **Why?**
  - **OK to do further computations with ∞ E.g., X/0 > Y may be a valid comparison**
  - **Ask math majors**
- **IEEE 754 represents ± ∞**
  - **Most positive exponent reserved for ∞**
  - **Significands all zeroes**

Representation for 0

- **Represent 0?**
  - **exponent all zeroes**
  - **significand all zeroes**
  - **What about sign? Both cases valid.**

```
+0: 0 00000000 00000000000000000000000
-0: 1 00000000 00000000000000000000000
```

Special Numbers

- What have we defined so far? (Single Precision)

Exponent	Significand	Object
0	0	0
0	nonzero	???
1-254	anything	+/- fl. pt. #
255	0	+/- ∞
255	nonzero	???

- Professor Kahan had clever ideas; "Waste not, want not"
  - We'll talk about Exp=0,255 & Sig!=0 later

Representation for Not a Number

- What do I get if I calculate `sqrt(-4.0)` or `0/0`?
  - If ∞ not an error, these shouldn't be either
  - Called Not a Number (NaN)
  - Exponent = 255, Significand nonzero
- Why is this useful?
  - Hope NaNs help with debugging?
  - They contaminate: op(NaN, X) = NaN

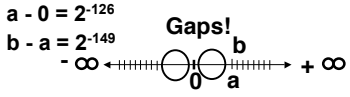
Representation for Denorms (1/2)

- Problem: There's a gap among representable FP numbers around 0
  - Smallest representable pos num:
    $a = 1.0\ldots_2 * 2^{-126} = 2^{-126}$
  - Second smallest representable pos num:
    $b = 1.000\ldots\ldots1_2 * 2^{-126}$
    $= (1 + 0.00\ldots1_2) * 2^{-126}$
    $= (1 + 2^{-23}) * 2^{-126}$
    $= 2^{-126} + 2^{-149}$

Normalization and implicit 1 is to blame!

$a - 0 = 2^{-126}$

$b - a = 2^{-149}$



Representation for Denorm (2/2)

- Solution:
  - We still haven't used Exponent=0, Significand nonzero
  - Denormalized number: no (implied) leading 1, implicit exponent = -126
  - Smallest representable pos num:
    - $A = 2^{-149}$
  - Second smallest representable pos num:
    - $b = 2^{-148}$

Special Numbers Summary

- Reserve exponents, significands:

Exponent	Significand	Object
0	0	0
0	nonzero	Denorm
1-254	Anything	+/- fl. Pt #
255	0	+/- ∞
255	nonzero	NaN

Rounding

- When we perform math on real numbers, we have to worry about rounding to fit the result in the significant field.
- The FP hardware carries two extra bits of precision, and then round to get the proper value
- Rounding also occurs when converting:
  double to a single precision value, or
  floating point number to an integer

IEEE FP Rounding Modes

Examples in decimal (but, of course, IEEE754 in binary)

- Round towards + ∞
  - ALWAYS round "up": 2.001 → 3, -2.001 → -2
- Round towards - ∞
  - ALWAYS round "down": 1.999 → 1, -1.999 → -2
- Truncate
  - Just drop the last bits (round towards 0)
- Unbiased (default mode). Midway? Round to even
  - Normal rounding, almost: 2.4 → 2, 2.6 → 3, 2.5 → 2, 3.5 → 4
  - Round like you learned in grade school (nearest int)
  - Except if the value is right on the borderline, in which case we round to the nearest EVEN number
  - Insures fairness on calculation
  - This way, half the time we round up on tie, the other half time we round down. Tends to balance out inaccuracies

Peer Instruction

1	1000 0001	111 0000 0000 0000 0000 0000

What is the decimal equivalent of the floating pt # above?

```
1: -1.75
2: -3.5
3: -3.75
4: -7
5: -7.5
6: -15
7: -7 * 2^129
8: -129 * 2^7
```

Peer Instruction Answer

What is the decimal equivalent of:

1	1000 0001	111 0000 0000 0000 0000 0000
S	Exponent	Significand

$(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$

$(-1)^1 \times (1 + .111) \times 2^{(129-127)}$

$-1 \times (1.111) \times 2^{(2)}$

-111.1

-7.5

```
1: -1.75
2: -3.5
3: -3.75
4: -7
5: -7.5
6: -15
7: -7 * 2^129
8: -129 * 2^7
```


Peer Instruction

1. Converting `float` -> `int` -> `float` produces same `float` number
2. Converting `int` -> `float` -> `int` produces same `int` number
3. FP add is associative:
   `(x+y)+z = x+(y+z)`

```
   ABC
1: FFF
2: FFT
3: FTF
4: FTT
5: TFF
6: TFT
7: TTF
8: TTT
```

Peer Instruction Answer

1. Converting `float` -> `int` -> `float` produces same `float` number — **FALSE**
2. Converting `int` -> `float` -> `int` produces same `int` number — **FALSE**
3. FP add is associative: `(x+y)+z = x+(y+z)` — **FALSE**

1. 3.14 -> 3 -> 3
2. 32 bits for signed int, but 24 for FP mantissa?
3. x = biggest pos #, y = -x, z = 1 (x != inf)

```
   ABC
1: FFF
2: FFT
3: FTF
4: FTT
5: TFF
6: TFT
7: TTF
8: TTT
```

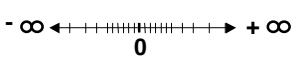
Peer Instruction

- Let `f(1,2)` = # of floats between 1 and 2
- Let `f(2,3)` = # of floats between 2 and 3

```
1: f(1,2) < f(2,3)
2: f(1,2) = f(2,3)
3: f(1,2) > f(2,3)
```

Peer Instruction Answer

- Let `f(1,2)` = # of floats between 1 and 2
- Let `f(2,3)` = # of floats between 2 and 3




```
1: f(1,2) < f(2,3)
2: f(1,2) = f(2,3)
3: f(1,2) > f(2,3)
```


"And in conclusion…"

- Reserve exponents, significands:

Exponent	Significand	Object
0	0	0
0	nonzero	Denorm
1-254	Anything	+/- fl. Pt #
255	0	+/- ∞
255	nonzero	NaN

- 4 Rounding modes (default: unbiased)
- MIPS FI ops complicated, expensive

Bonus slides

- These are extra slides that used to be included in lecture notes, but have been moved to this, the "bonus" area to serve as a supplement.
- The slides will appear in the order they would have in the normal presentation

Bonus

FP Addition

- More difficult than with integers
- Can't just add significands
- How do we do it?
  - De-normalize to match exponents
  - Add significands to get resulting one
  - Keep the same exponent
  - Normalize (possibly changing exponent)
- Note: If signs differ, just perform a subtract instead.

MIPS Floating Point Architecture (1/4)

- MIPS has special instructions for floating point operations:
  - Single Precision:
    `add.s, sub.s, mul.s, div.s`
  - Double Precision:
    `add.d, sub.d, mul.d, div.d`
- These instructions are far more complicated than their integer counterparts. They require special hardware and usually they can take much longer to compute.

MIPS Floating Point Architecture (2/4)

- Problems:
  - It's inefficient to have different instructions take vastly differing amounts of time.
  - Generally, a particular piece of data will not change from FP to int, or vice versa, within a program. So only one type of instruction will be used on it.
  - Some programs do no floating point calculations
  - It takes lots of hardware relative to integers to do Floating Point fast

MIPS Floating Point Architecture (3/4)

- **1990 Solution: Make a completely separate chip that handles only FP.**
- **Coprocessor 1: FP chip**
  - **contains 32 32-bit registers: `$f0`, `$f1`, …**
  - **most registers specified in `.s` and `.d` instruction refer to this set**
  - **separate load and store: `lwc1` and `swc1` ("load word coprocessor 1", "store …")**
  - **Double Precision: by convention, even/odd pair contain one DP FP number: `$f0`/`$f1`, `$f2`/`$f3`, … , `$f30`/`$f31`**

MIPS Floating Point Architecture (4/4)

- **1990 Computer actually contains multiple separate chips:**
  - **Processor: handles all the normal stuff**
  - **Coprocessor 1: handles FP and only FP;**
  - **more coprocessors?… Yes, later**
  - **Today, cheap chips may leave out FP HW**
- **Instructions to move data between main processor and coprocessors:**
  - **`mfc0`, `mtc0`, `mfc1`, `mtc1`, etc.**
- **Appendix pages A-70 to A-74 contain many, many more FP operations.**

Example: Representing 1/3 in MIPS

- **1/3**
  = $0.33333\ldots_{10}$
  = 0.25 + 0.0625 + 0.015625 + 0.00390625 + …
  = 1/4 + 1/16 + 1/64 + 1/256 + …
  = $2^{-2} + 2^{-4} + 2^{-6} + 2^{-8} + \ldots$
  = $0.0101010101\ldots_2 * 2^0$
  = $1.0101010101\ldots_2 * 2^{-2}$
  - **Sign: 0**
  - **Exponent = -2 + 127 = 125 = 01111101**
  - **Significand = 0101010101…**

0	0111 1101	0101 0101 0101 0101 0101 010

Casting floats to ints and vice versa

`(int)` *`floating_point_expression`*
Coerces and converts it to the nearest integer (C uses truncation)

```
i = (int) (3.14159 * f);
```

`(float)` *`integer_expression`*
converts integer to nearest floating point

```
f = f + (float) i;
```

int → float → int

```
if (i == (int)((float) i)) {
 printf("true");
}
```

- **Will not always print "true"**
- **Most large values of integers don't have exact floating point representations!**
- **What about `double`?**

float → int → float

```
if (f == (float)((int) f)) {
 printf("true");
}
```

- **Will not always print "true"**
- **Small floating point numbers (<1) don't have integer representations**
- **For other numbers, rounding errors**

Floating Point Fallacy

- **FP add associative: FALSE!**
  - **x = – 1.5 x 10^{38}, y = 1.5 x 10^{38}, and z = 1.0**
  - **x + (y + z) = –1.5x10^{38} + (1.5x10^{38} + 1.0) = –1.5x10^{38} + (1.5x10^{38}) = 0.0**
  - **(x + y) + z = (–1.5x10^{38} + 1.5x10^{38}) + 1.0 = (0.0) + 1.0 = 1.0**
- **Therefore, Floating Point add is not associative!**
  - **Why? FP result approximates real result!**
  - **This example: 1.5 x 10^{38} is so much larger than 1.0 that 1.5 x 10^{38} + 1.0 in floating point representation is still 1.5 x 10^{38}**