CS 70 FALL 2006 — DISCUSSION #12

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1. Administrivia

(1) Course Information
   - Homework #11 is due on Monday.

2. Central Limit Theorem

Exercise 1. You roll 420 standard dice. Use the approximation given by the Central Limit Theorem to answer the following questions. Can we be sure that that their sum will be less than 1500 at a 84,4% confidence level? In which interval should the sum fall at a 98% confidence level?

Exercise 2. Compare the bound given by Chebyshev Inequality and that given by the Central Limit Theorem for a variable $X = X_1 + X_2 + X_3 + X_n$, where the $\{X_i\}$'s are i.i.d with mean μ and variance σ and n is large. How much better is the CLT bound for a 1σ variation? For a 2σ variation?

3. Estimates - Harmonic Numbers

The nth harmonic number H_n was defined today in class as:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

We proved that $H_n \approx \log n$ (can you recall the proof?) and used this fact to analyze the Coupon Collector process.

Exercise 3. Top-in-at-random shuffle. Consider the following reasonable way of shuffling a deck of card. At every iteration pick a random position in the deck and insert the top card in this position. How many iterations will it take on average for the card originally at the bottom to reach the top? Would you keep shuffling once this happens?

4. Estimates - The factorial function

Today in class we saw the following simple yet powerful bound:

$$1 + x \leq e^x$$

And we noticed how we can use it to derive the inequality:

$$\left(1 + \frac{1}{n}\right)^n \leq e$$

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Now, we are going to apply this in deriving a good approximation to the factorial function:

Theorem 1. *For every $n \geq 1$, we have:*

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n$$

Proof. We start with the upper bound. The proof is by induction on n. The claim holds for $n = 1$. Suppose it holds for $n = j - 1$. Then;

$$\begin{aligned} j! &= j \cdot (j-1) \leq je(j-1)\left(\frac{j-1}{e}\right)^{j-1} = \\ &= \left[ej\left(\frac{j}{e}\right)^j\right]\left(\frac{j-1}{j}\right)^j e \end{aligned}$$

Exercise 4. Apply the bound above to complete the proof

Exercise 5. Prove the lower bound in a very similar fashion □

Exercise 6. Use the Theorem to give approximations for the binomial coefficents