EECS 70 Discrete Mathematics and Probability Theory
Fall 2014 Anant Sahai

Discussion 13M

1. Chebyshev's Inequality

Derive the Chebyshev's Inequality $\Pr[|X-\mu| \geq \alpha] \leq \frac{\mathrm{Var}(X)}{\alpha}$ using Markov Inequality $\Pr[X \geq \alpha] \leq \frac{\mathbf{E}[X]}{\alpha}$.

2. Working with the law of large numbers

(a) A fair coin is tossed and you win a prize if there are more than 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed and you win a prize if there are more than 40% heads. Which is better: 10 tosses or 100 tosses? Explain.

(c) A coin is tossed and you win a prize if there are between 40% and 60% heads. Which is better: 10 tosses or 100 tosses? Explain.

(d) A coin is tossed and you win a prize if there are exactly 50% heads. Which is better: 10 tosses or 100 tosses? Explain.

3. Tellers

Imagine that X is the number of customers that enter a bank at a given hour. To simplify everything, in order to serve n customers you need at least n tellers. One less teller and you won't finish serving all of the customers by the end of the hour. You are the manager of the bank and you need to decide how many tellers there should be in your bank so that you finish serving all of the customers in time. You need to be sure that you finish in time with probability at least 95%.

(a) Assume that from historical data you have found out that $\mathbf{E}[X] = 5$. How many tellers should you have?

(b) Now assume that you have also found out that $\mathrm{Var}(X) = 5$. Now how many tellers do you need?

4. Homework Polling

Suppose Prof. Sahai wants to poll the CS 70 students about whether the homeworks have been too hard recently. Suppose everyone is comfortable enough to answer honestly, either no (0) or yes (1). Let the true fraction of students who think the homework is too hard be p. Let the response of the i^{th} polled student be X_i. Prof. Sahai would like to poll n students, with n large enough that the probability of estimating p to within $\pm 2\%$ is at least 90%.

a) Let $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the average of the n responses. What is the expectation of M_n?
What is the event that we are interested in whose probability we would like to be at least 90%? Draw a picture of the distribution of $M_n - p$ and mark the region that corresponds to the event of interest.

b) Now use Chebyshev's inequality to find a safe n regardless of what p is.

c) What if instead of wanting an accuracy of $\pm 2\%$ we wanted a relative error of 2%. This means that if the true value was p, we want the answer we get to be within $[0.98p, 1.02p]$. Can we pull that off using Chebyshev's inequality? Do you think we could pull that off with a single universal choice of n that does not depend on (the unknown) p?

5. Will I Get My Package?

Sneaky delivery guy of some company is out delivering n packages to n customers. Not only does he hand a random package to each customer, he tends to open a package before delivering with probability $\frac{1}{2}$. Let X be the number of customers who receive their own packages unopened. Compute the expectation $\mathrm{E}(X)$ and the variance Var(X).