1. **Two Color Theorem**

   Consider a scenario where we have a 2D plane that we divide into regions by drawing straight lines. Using induction, prove that we can color this map using no more than two colors such that no two regions that share a boundary have the same color.

2. **Power Inequality**

   Prove that when n is a positive integer greater than 1, $2^n + 3^n < 5^n$.

3. **Bit String**

   Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \ldots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

   where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

4. Grid Induction A bug is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes). Every second he does one of the following (if possible):

(i) Jump one inch down, to $(i, j-1)$.

(ii) Jump one inch left, to $(i-1, j)$.

For example, if he is at $(5, 0)$, his only option is to jump left to $(4, 0)$.

Prove that no matter how he jumps, he will always reach $(0, 0)$ in finite time.

5. Summations

Prove by induction that the following formulas hold for any natural number n.

1. $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

2. $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

6. Proofs, Perhaps

The proofs below are INCORRECT! Explain *clearly* and *concisely* where the logical error in the proof lies.

1.

Claim: $(\forall n \in \mathbb{N})(n^2 \leq n)$
Proof: Base Case: When $n = 1$, the statement is $1^2 \leq 1$ which is true.
Induction hypothesis: Assume that $k^2 \leq k$.
Inductive step: We need to show that

$$(k+1)^2 \leq k+1.$$

Working backwards we see that:

$$k^2 \leq (k+1)^2 - 1 \leq (k+1) - 1 = k.$$

So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in \mathbb{N}$ we know that $n^2 \leq n$.

2.

Claim: $(\forall n \in \mathbb{N})(7^n = 1)$
Proof: (uses strong induction)
Base Case: Certainly $7^0 = 1$.
Induction hypothesis: Assume that $7^j = 1$ for all $0 \leq j \leq k$.
Inductive step: We need to prove that $7^{k+1} = 1$. But,

$$7^{k+1} = \frac{(7^k . 7^k)}{7^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

Hence, by the Principle of Strong Induction, $7^n = 1$ for all $n \in \mathbb{N}$.

3.

Claim: All natural numbers are equal.
Proof: It is sufficient to show that for any two natural numbers a and b, $a = b$. Further, it is sufficient to show that for all $n \geq 0$, if a and b satisfy $\max\{a, b\} = n$ then $a = b$. We proceed by induction on n.
Base case: If $n = 0$ then a and b, being natural numbers, must both be 0. So clearly $a = b$.
Inductive step: Assume that the claim is true for some value n. Take a and b with $\max\{a, b\} = n + 1$. Then $\max\{a - 1, b - 1\} = n$, and hence by the induction hypothesis $a - 1 = b - 1$. Consequently, $a = b$.