Discussion 2W

1. **More squares**
   Prove the following statement: $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $\sum_{k=0}^{n} k^3 = m^2$

2. **Proving Inequality**
   For all positive integers $n \geq 1$, prove that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}$$

3. Well-Ordered Grid

Consider an infinite sheet of graph paper such that each square contains a natural number. Suppose that the number in each square is equal to the average of the numbers in the four neighboring squares.

(a) By the Well-Ordering Principle, there must be some smallest number in the grid (call it n). Prove that for any square containing n, the four squares adjacent to it must also contain n.

(b) Prove that each square in the infinite grid contains the same number.

4. $\sqrt{2}$ is irrational

A rational number is any number that can be expressed as a fraction $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.

(a) Is the fraction expression $\frac{p}{q}$ for a rational number unique?

(b) Prove that $\sqrt{2}$ is irrational using well-ordering principle.