Discussion 3W

1. True or False

For each of the following claims, state whether the claim is true or false. If it is true, give a *short* proof; if it is false, give a *simple* counterexample.

(a) In a stable marriage instance, if man m and woman w each put each other at the top of their respective preference lists, then m must be paired with w in every stable matching.

(b) In a stable marriage instance with at least two men and two women, if man m and woman w each put each other at the bottom of their respective preference lists, then m cannot be paired with w in any stable pairing.

2. Propose-and-Reject Proofs

Prove the following statements about the traditional propose-and-reject algorithm.

(a) In any execution of the algorithm, if a woman receives a proposal on day i, then she receives some proposal on every day thereafter until termination.

(b) In any execution of the algorithm that takes k days, there must be some woman who does not receive a proposal in day $k-1$.

(c) In any execution of the algorithm, if a woman receives no proposal on day i, then she receives no proposal on any previous day j, $1 \leq j < i$.

(d) In any execution of the algorithm, there is at least one woman who only receives a single proposal (Hint: use the parts above!)

3. Good, Better, Best

In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable matchings, M_1, M_2, and M_3. Also, each man m has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man m_1, this order is $M_1 > M_2 > M_3$.

Prove that every man has the same preference ordering $M_1 > M_2 > M_3$.

4. Pairing Up

Prove that for every even $n \geq 2$, there exists an instance of the stable marriage problem with n men and n women such that the instance has at least $2^{n/2}$ distinct stable matchings.