EECS 70 Discrete Mathematics and Probability Theory
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Discussion 8A

Hi, all! Please do not feel all the problems need be done in section. We added a couple to allow for extra practice as some students requested this in the midterm check-in survey. Enjoy!

0. Warmup: Clothes and stuff

1. Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, obvi). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

2. It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

3. If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

4. We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters, because no one really wants to see that burgundy one next to our favorite forest green fedora.)

5. Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of a, your answer from the previous part.)

6. Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

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1. Counting

1. How many ways are there to arrange n 1s and k 0s into a sequence?

2. How many solutions does
   $x_0 + x_1 + \ldots + x_k = n$
   have, if all xs must be non-negative integers?

3. How many solutions does
   $x_0 + x_1 = n$
   have, if all xs must be *strictly positive* integers?

4. How many solutions does
   $x_0 + x_1 + \ldots + x_k = n$
   have, if all xs must be *strictly positive* integers?

2. Fermat's necklace

Let p be a prime number and let k be a positive integer. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable.

1. We have a piece of string. As a relaxing study break, we want to make a pretty garland by threading p beads onto the string. How many different ways are there construct such a sequence of p beads of k different colors?

2. Now let's add a restriction. We want our garland to be exciting and multicolored. Now how many different sequences exist? (Your answer should be a simple function of k and p.)

3. Now we tie the two ends of the string together, forming a circular necklace which lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

   How many non-equivalent sequences are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

   [Hint: What follows if rotating all the beads on a necklace to another position produces an identical looking necklace?]

4. Use your answer to part (c) to prove Fermat's little theorem. (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)

3. Story Problems

Prove the following identities by combinatorial argument:

1. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

2. $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

3. $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$

4. Countability Basics

1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (one-to-one)? Briefly justify.

2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ a surjection (onto)? Briefly justify.