EECS 70 Discrete Mathematics and Probability Theory
Fall 2015 Jean Walrand Discussion 10b

1. Unlikely events

1. Toss a fair coin x times. What is the probability that you never get heads?
   Solution: 0.5^x

2. Roll a fair die x times. What is the probability that you never roll a six?
   Solution: $(1-\frac{1}{6})^x$

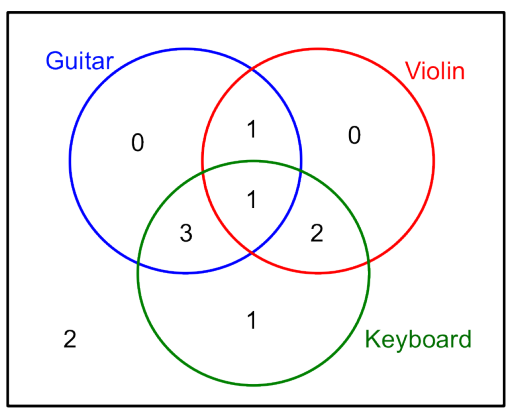
3. Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?
   Solution: $(1-1/10^6)^x$

4. How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?
   Solution: For coin, want: $0.5^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 0.5} \approx 3.32$
   For die, want: $(5/6)^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 5/6} \approx 12.6$
   For coin, want: $(1-1/10^6)^x \leq 0.1$ so $x \geq \frac{\log 0.1}{\log 1-1/10^6} \approx 2*10^6$ Comment on how answer for coin is almost exactly equal to $\log 0.1/(1/10^6)$ using the approximation $(1-x) \approx e^{(-x)}, xbeing 1/10^6$

2. Random Band

In a group of ten people, seven can play the keyboard, five can play the guitar, four can play the violin, four can play the keyboard and the guitar, three can play the keyboard and the violin, two can play the guitar and the violin, and one person can play all three instruments.

1. Suppose a person is picked uniformly at random from the group. Draw a Venn Diagram and use it to calculate the probability that this person can play at least one instrument.
   Solution:



Therefore the probability a randomly chosen person can play at least one instrument is $\frac{8}{10} = \frac{4}{5}$.

2. Let $P(K)$ be the probability a randomly chosen person plays the keyboard, $P(G)$ be the probability a randomly chosen person plays the guitar, and $P(V)$ be the probability that a randomly chosen person plays the violin. Use the Inclusion-Exclusion principle to calculate the same probability as in part (a).
Solution:

$$P(K)+P(G)+P(V)-P(K,G)-P(K,V)-P(G,V)+P(K,G,V)$$
$$=\frac{7}{10}+\frac{5}{10}+\frac{4}{10}-\frac{4}{10}-\frac{3}{10}-\frac{2}{10}+\frac{1}{10}=\frac{8}{10}=\frac{4}{5}$$

3. Birthdays

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

1. what is the probability that the first 3 people do not have the same birthday?
Solution: $364/365 * 363/365$

2. what is the probability that the first three people have the same birthday?
Solution: $(1/365)^2$

3. What is the probability that it takes more than 20 people for this to occur?
Solution: $\Pr[\text{it takes more than 20 people}] = \Pr[\text{20 people don't have the same birthday}] = \frac{\frac{365!}{(365-20)!}}{365^{20}} = \boxed{\frac{\frac{365!}{345!}}{365^{20}}} \approx .589$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes more than 20 people}]\\
=&\Pr[b_{20}\neq b_i \mid b_i\text{'s are all different } \forall 1\leq i\leq 19]\times\Pr[b_i\text{'s are all different } \forall 1\leq i\leq 19]\\
=&\Pr[b_{20}\neq b_i \mid b_i\text{'s are all different } \forall 1\leq i\leq 19]\times\Pr[b_{19}\neq b_i \mid b_i\text{'s are all different } \forall 1\leq i\leq 18]\times\\
&\quad\cdots\times\Pr[b_3\neq b_i \mid b_i\text{'s are all different } \forall 1\leq i\leq 2]\times\Pr[b_2\neq b_1]\\
=&\frac{365-19}{365}\times\frac{365-18}{365}\times\cdots\times\frac{363}{365}\times\frac{364}{365}\\
\approx&.589
\end{aligned}$$

4. What is the probability that it takes exactly 20 people for this to occur?
Solution: $\Pr[\text{it takes exactly 20 people}] =$
$\Pr[\text{first 19 have different birthdays } \textbf{and } 20^{\text{th}}\text{person shares a birthday with one of the first 19}]$.
How total ways can the birthdays be chosen for 20 people? 365^{20}. How many ways can the birthdays be chosen so the first 19 have different birthdays and the 20th person shares a birthday with the first 19? Well, the firs person has 365 choices, the second has 364 choices left, and so on until the nineteenth person has $(365-19+1)=347$ choices left. Then, the 20th person has 19 choices for his birthday. So in total, there are $365\cdot 364\cdot\cdots\cdot 348\cdot 347\cdot 19=\frac{365!}{346!}\cdot 19$ ways of getting what we want. So $\Pr[\text{it takes exactly 20 people}] = \frac{365\cdot 364\cdot\cdots\cdot 348\cdot 347\cdot 19}{365^{20}} = \boxed{\frac{\frac{365!}{346!}\cdot 19}{365^{20}}} \approx .032$

Another explanation that does not use counting:
Let b_i be the birthday of the i-th person.

$$\begin{aligned}
&\Pr[\text{it takes exactly 20 people}]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \le i \le 19] \times \Pr[b_i\text{'s are all different } \forall 1 \le i \le 19]\\
=&\Pr[b_{20} \text{ is equal to one of the } b_i\text{'s} \mid b_i\text{'s are all different } \forall 1 \le i \le 19] \times\\
&\Pr[b_{19} \ne b_i \mid b_i\text{'s are all different } \forall 1 \le i \le 18] \times \cdots \times \Pr[b_3 \ne b_i \mid b_i\text{'s are all different } \forall 1 \le i \le 2] \times \Pr[b_2 \ne b_1]\\
=&\frac{19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365}\\
\approx&.032
\end{aligned}$$

5. Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?
Solution: $\Pr[\text{it takes exactly 20 people}] =$
$\Pr[\text{first 19 don't have your birthday } \textbf{and } 20^{\text{th}} \text{person has your birthday}]$.
Similar to the last problem, there are 364 choices for the first person's birthday to be different than yours, 364 for the second person, and so on until the nineteenth person has 364 choices. Then, the 20th person has exactly 1 choice to have your birthday. So the total number of ways to get what we want is $364^{19} \cdot 1$. There are 365^{20} possibilities total. So $\Pr[\text{it takes exactly 20 people}] = \boxed{\frac{364^{19}}{365^{20}}} \approx .0026$

Another explanation that does not use counting:

$$\begin{aligned}
\Pr[\text{it takes exactly 20 people}] =&\Pr[\text{the 1st person does not have the same birthday as yours}] \times\\
&\Pr[\text{the 2nd person does not have the same birthday as yours}] \times\\
&\cdots \times \Pr[\text{the 19th person does not have the same birthday as yours}] \times\\
&\Pr[\text{the 20th person has the same birthday as yours}]\\
=&\frac{364}{365} \times \frac{364}{365} \times \cdots \frac{364}{365} \times \frac{1}{365}\\
=&\frac{364^{19} \times 1}{365^{20}}\\
\approx&0.0026
\end{aligned}$$

4. Throwing balls into a depth-limited bin

Say you want to throw n **distinct** balls into n bins with depth $k-1$ (they can fit $k-1$ balls, after that the bins overflow). Suppose that n is a large number and $k = 0.1n$. You throw the balls randomly into the bins, but you would like it if they don't overflow. You feel that you might expect not too many balls to land in each bin, but you're not sure, so you decide to investigate the probability of a bin overflowing.

1. Focus on the first bin. Get an upper bound the number of ways that you can throw the balls into the bins such that this bin overflows. Try giving an argument about the following strategy: select k balls to put in the first bin, and then throw the remaining balls randomly.
Solution: We choose k of the balls to throw in the first bin and then throw the remaining $n-k$, giving us $\binom{n}{k}n^{n-k}$. Certainly any outcome of the ball-throwing that overflows the first bin is accounted for –

we can simply choose the first k balls that land in the first bin and then simulate the rest of the outcome via random throwing. However, we are potentially overcounting: if $k+1$ balls go in the first bin, we have many choices for which k of them that could have been the "chosen" ones, and we count each one of these choices as distinct. However, they correspond to the same configuration, namely the one where $k+1$ balls are in the first bin. Hence we get an upper bound.

2. Calculate an upper bound on the probability that the first bin will overflow:

   Solution: We divide by the total number of ways the balls could have fallen into the bins, with order, so we get $\frac{\binom{n}{k}n^{n-k}}{n^n} = \frac{\binom{n}{k}}{n^k}$.

3. Upper bound the probability that some bin will overflow.

   Hint: for any events $A_1 ... A_m$, $P(A_1 \cup A_2 \cup ... A_m) \leq P(A_1) + P(A_2) + ... P(A_m)$. This is known as the union bound.

   Solution: By symmetry, we can just upper bound this probability by n multiplied by the probability that a single bin (wlog, the first bin) overflows. This gives about $n * \frac{\binom{n}{k}}{n^k}$. This technique is called a *union bound*, where we upper bound the probability of the union of a bunch of events by the sums of the probabilities of the events.

4. How does the above probability scale as n gets really large?

   Solution: We get $n * \frac{\binom{n}{k}}{n^k} = n * \frac{n*(n-1)*...*(n-k+1)}{k!n^k} \leq n * \frac{n^k}{k!n^k} = \frac{n}{k!} = \frac{n}{(0.1n)*(k-1)!} = \frac{10}{(0.1n-1)!}$. Clearly, as n gets large this probability is going to 0. Note that this same analysis would work with $k = cn$ for any constant $0 < c < 1$. Hence, using some very coarse upper bounds we can see that as the number of balls and bins grows we have that it is very unlikely that we get a constant fraction of the balls in any single bin.