## Lecture 12.

It's Friday.

Will let out early today!

61A midterm makeup afterwards.

Today: A bit of review, RSA, signature schemes.

## Inverses?

When does $a$ have inverse $\pmod{m}$?

When $gcd(a,m) = 1$!

## Proof review.

**Claim:** $a^{-1} \pmod{m}$ exists when $\gcd(a,m) = 1$.

Fact: $ax \neq ay \pmod{m}$ for $x \neq y \in \{0, \ldots m-1\}$

Proof of Fact: Let $ax = ay \pmod{m}$, $x \neq y \in \{0, \ldots, m-1\}$
$ax = ay + km$
$a(x-y) = km$
Consider prime factorization:
$a = a_1 \ldots a_\ell$.
$m = m_1 \ldots m_z$.
Do any $a_i = m_j$? Yes? No? No! $\gcd(a,m) = 1$!
Therefore $a(x-y) = km$
only if factorization of $(x-y)$ contains all factors of $m$.
$\implies (x-y) \geq m$ or $(x-y) = 0$. Contradiction. □

## Excursion: Bijections.

$f : S \to T$ is **one-to-one mapping**.
one-to-one: $f(x) \neq f(x')$ for $x, x' \in S$ and $x \neq x'$. Not 2 to 1!
$f(\cdot)$ is **onto**
if for every $y \in T$ there is $x \in S$ where $y = f(x)$.

Bijection is one-to-one and onto function.
Two sets have the same size
if and only if there is a bijection between them!

Same size?
$\{red, yellow, blue\}$ and $\{1,2,3\}$?
$f(red) = 1$, $f(yellow) = 2$, $f(blue) = 3$.
$\{red, yellow, blue\}$ and $\{1,2\}$?
$f(red) = 1$, $f(yellow) = 2$, $f(blue) = 2$.
two to one! not one to one.
$\{red, yellow\}$ and $\{1,2,3\}$?
$f(red) = 1$, $f(yellow) = 2$.
Misses 3. not onto.

## Modular arithmetic examples.

$f : S \to T$ is **one-to-one mapping**.
one-to-one: $f(x) \neq f(x')$ for $x, x' \in S$ and $x \neq y$.
$f(\cdot)$ is **onto**
if for every $y \in T$ there is $x \in S$ where $y = f(x)$.

Recall: $f(red) = 1$, $f(yellow) = 2$ , $f(blue) = 3$
One-to-one if inverse: $g(1) = red$, $g(2) = yellow$, $g(3) = blue$.

Is $f(x) = x + 1 \pmod{m}$ one-to-one? $g(x) = x - 1 \pmod{m}$.
Onto: range is subset of domain.
Is $f(x) = ax \pmod{m}$ one-to-one?
If $\gcd(a,m) = 1$, $ax \neq ax' \pmod{m}$.

Injective? Surjective?
We tend to use one-to-one and onto.

**Bijection** is one-to-one and onto function.
Two sets have the same size
if and only if there is a bijection between them!

## Inverses: continued.

**Claim:** $a^{-1} \pmod{m}$ exists when $\gcd(a,m) = 1$.

Fact: $ax \neq ay \pmod{m}$ for $x \neq y \in \{0, \ldots m-1\}$

Consider $T = \{0a \pmod{m}, 1a \pmod{m}, \ldots, \ldots (m-1)a \pmod{m}\}$
Consider $S = \{0, 1, \ldots, \ldots (m-1)\}$
$S = T$. Why?
$T \subseteq S$ since $ax \pmod{m|} \in \{0, \ldots, m-1\}$
One-to-one mapping from $S$ to $T$!
$\implies |T| \geq |S|$
Same set.

Why does $a$ have inverse? $T$ is $S$ and therefore contains 1 ! ! !

Why am I excited? There is an $x$ where $ax = 1$.
There is an inverse of $a$! ! ! □

## Fermat from Bijection.

**Fermat's Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod p$,

$$a^{p-1} \equiv 1 \pmod p.$$

**Proof:** Consider $T = \{a \cdot 1 \pmod p, \ldots, a \cdot (p-1) \pmod p\}$.

$T$ is range of function $f(x) = ax \mod (p)$ for set $S = \{1, \ldots, p-1\}$.
Invertible function: one-to-one.
$T \subseteq S$ since $0 \notin T$.
$p$ is prime.
$\implies T = S$.
Product of elts of $T$ = Product of elts of $S$.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

Each of $2, \ldots (p-1)$ has an inverse modulo $p$, mulitply by inverses to get...

$$a^{(p-1)} \equiv 1 \mod p.$$

□

## RSA and Fermat.

RSA:

Alice:
Primes: $p$, $q$. $N = pq$.
Encryption Key: $e$ where $\gcd(e, (p-1)(q-1))) = 1$
Decryption Key: $d = e^{-1} \pmod{(p-1)(q-1)}$

Message: $m$
Encryption: $y = E(m) = m^e \pmod N$. Bob.
Decryption: $D(y) = y^d \pmod N$. Alice.
Result: $m^{ed} \pmod N$

Example:

$p = 7, 1 = 11$. $N = 77$
$e = 7 \quad \gcd(7, 60) = 1$.
$d = 43 \quad 7 * 43 = 1 \pmod{60}$.

$x = 2$
$y = 2^7 \pmod{77}$ Bob.
$y^{43} = 2 \pmod{77}$ Alice.
Alice got Bob's message!

Want $D(E(x)) = x$
**Thm:** $x^{ed} = x \pmod N$

Alice got message back!!!

## Fermat: a seeming excursion?

**Thm:** $m^{ed} = m \pmod{pq}$ if $ed = 1 \pmod{(p-1)(q-1)}$
Seems like magic!

**Fermat's Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod p$,

$$a^{p-1} \equiv 1 \pmod p.$$

$3^6 \pmod 7$? 1.
$3^7 \pmod 7$? 3.

Involves exponents and gets 3 back.
Seems like magic.

**Corollary:** $a^{k(p-1)+1} = a \pmod p$

Get $a$ back when exponent is $1 \pmod{p-1}$.
A little like RSA.
$a^{ed} \pmod{(p-1)(q-1)}$ is $a$
when exponent is $1 \pmod{(p-1)(q-1)}$.

**Proof of Corollary.** If $a = 0$, $a^{k(p-1)+1} = 0 \pmod m$.
Otherwise $a^{1+k(p-1)} \equiv a^1 * (a^{p-1})^k \equiv a * (1)^b \equiv a \pmod p$ □

Idea: Fermat removes the $k(p-1)$ from the exponent!

## ...Decoding correctness...

**Lemma 1:** For any prime $p$ and any $a, b$,
$a^{1+b(p-1)} \equiv a \pmod p$

**Lemma 2:** For any two different primes $p, q$ and any $x, k$,
$x^{1+k(p-1)(q-1)} \equiv x \pmod{pq}$

Let $a = x$, $b = k(p-1)$ and apply Lemma 1 with modulus $q$.

$$x^{1+k(p-1)(q-1)} \equiv x \pmod q$$

$x^{1+k(q-1)(p-1)} - x \equiv 0 \mod (q) \implies$ multiplie of $q$.

Let $a = x$, $b = k(q-1)$ and apply Lemma 1 with modulus $p$.

$$x^{1+k(p-1)(q-1)} \equiv x \pmod p$$

$x^{1+k(q-1)(p-1)} - x \equiv 0 \mod (p) \implies$ multiplie of $p$.

$x^{1+k(q-1)(p-1)} - x$ is multiple of $p$ and $q$.

$x^{1+k(q-1)(p-1)} - x \equiv 0 \mod (pq) \implies x^{1+k(q-1)(p-1)} = x \mod pq.$

□

## RSA decodes correctly..

**Lemma 2:** For any two different primes $p, q$ and any $x, k$,
$x^{1+k(p-1)(q-1)} \equiv x \pmod{pq}$

**Theorem:** RSA correctly decodes!
Recall

$$D(E(x)) = (x^e)^d = x^{ed} \equiv x \pmod{pq},$$

where $ed \equiv 1 \mod (p-1)(q-1) \implies ed = 1 + k(p-1)(q-1)$

$$x^{ed} \equiv x^{k(p-1)(q-1)+1} \equiv x \pmod{pq}.$$

□

## Construction of keys.. ..

1. Find large (100 digit) primes $p$ and $q$?
   **Prime Number Theorem:** $\pi(N)$ number of primes less than $N$.For all $N \geq 17$

   $$\pi(N) \geq N / \ln N.$$

   Choosing randomly gives approximately $1/(\ln N)$ chance of number being a prime. (How do you tell if it is prime? ... cs170..Miller-Rabin test.. Primes in $P$).

   For 1024 bit number, 1 in 710 is prime.
2. Choose $e$ with $\gcd(e, (p-1)(q-1)) = 1$.
   Use gcd algorithm to test.
3. Find inverse $d$ of $e$ modulo $(p-1)(q-1)$.
   Use extended gcd algorithm.

All steps are polynomial in $O(\log N)$, the number of bits.

## Security of RSA.

Security?

1. Alice knows $p$ and $q$.
2. Bob only knows, $N(= pq)$, and $e$.
   Does not know, for example, $d$ or factorization of $N$.
3. Breaking this scheme $\Longrightarrow$ factoring $N$.
   Don't know how to factor $N$.

## Much more to it.....

If Bobs sends a message (Credit Card Number) to Alice,

Eve sees it.

Eve can send credit card again!!

The protocols are built on RSA but more complicated;
For example, several rounds of challenge/response.

One trick:
Bob encodes credit card number, $c$,
concatenated with random $k$-bit number $r$.

Never sends just $c$.

Again, more work to do to get entire system.

CS161...