Lecture 14. Outline.

1. Finish Polynomials and Secrets.
2. Finite Fields: Abstract Algebra
3. Erasure Coding

Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact: There is exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p that contains $d+1$ pts.

Note: The points have to have different x values!

Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k shares gives secret.
Knowing k pts, find unique $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

There exists a polynomial...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Proof of at least one polynomial:
Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_{d+1}\Delta_{d+1}(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$. Degree d polynomial!

Construction proves the existence of a polynomial!

Reiterating Examples.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

Degree 1 polynomial, $P(x)$, that contains $(1,3)$ and $(3,4)$?

Work modulo 5.

$\Delta_1(x)$ contains $(1,1)$ and $(3,0)$.

$$\begin{aligned}\Delta_1(x) &= \frac{(x-3)}{1-3} = \frac{x-3}{-2} \\ &= 2(x-3) = 2x - 6 = 2x + 4 \pmod 5.\end{aligned}$$

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3); (2,4); (3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1); (2,0); (3,0)$.

$$\begin{aligned}\Delta_1(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = 3(x-2)(x-3) \\ &= 3x^2 + 1 \pmod 5\end{aligned}$$

Put the delta functions together.

Simultaneous Equations Method.

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1) + b &\equiv m + b \equiv 3 \pmod 5 \\ P(2) = m(2) + b &\equiv 2m + b \equiv 4 \pmod 5 \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m + b &\equiv 3 \pmod 5 \\ m &\equiv 1 \pmod 5 \end{aligned}$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2); (2,4); (3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

In general..

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Solve...

$$\begin{aligned} a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod p \\ a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod p \\ \vdots \qquad & \vdots \quad \vdots \\ a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod p \end{aligned}$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Uniqueness.

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Existence: Lagrange interpolation. Uniqueness?

Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

Uniqueness Fact. At most one degree d polynomial contains $d+1$ points.

Proof:

Roots fact: Any degree d polynomial has at most d roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hits $d+1$ points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree d.
Contradiction. □
Must prove **Roots fact.**

Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
                 4 x + 4 r 4
        -----------------
x - 3 ) 4x^2 - 3 x + 2
        4x^2 - 2x
        ----------
                 4x + 2
                 4x - 2
                 -------
                       4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod 5$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x-a)Q(x) + r$

r is degree 0 polynomial..or a constant!

Only d roots.

Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
$P(x) = (x-a)Q(x)$.

Proof: $P(x) = (x-a)Q(x) + r$.
Plugin a: $P(a) = r = 0$. □

Lemma 2: $P(x)$ has d roots; $r_1, \ldots, r_d$ then
$P(x) = (x-r_1)(x-r_2)\cdots(x-r_d)c(x)$.

Proof Sketch: By induction.

Base Case: degree 0. No roots.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1.
$Q(x)$ has smaller degree ...
so by induction hypothesis...
we are done. □

Thus,$d+1$ roots implies degree is at least $d+1$.

The contrapositive...

Roots fact: Any degree d polynomial has at most d roots.

Summary.

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Existence:
Lagrange Interpolation.

Uniqueness:
At most d roots for degree d polynomial.

Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime p has multiplicative inverses..

..and has only a finite number of elements.

Good for computer science.

Arithmetic modulo a prime p is a **finite field** denoted by F_p or $GF(p)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Secret Sharing

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

Shamir's k out of n Scheme:
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1. Choose $a_0 = s$, and random $a_1,\ldots,a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.
Efficiency: ???

Efficiency.

Need $p > n$ to hand out n shares: $P(1)\ldots P(n)$.

For b-bit secret, must choose a prime $p > 2^b$.

Theorem: There is always a prime between n and $2n$.

Working over numbers within 1 bit of secret size.
Minimal!

With k shares, reconstruct polynomial, $P(x)$.

With $k-1$ shares, any of p values possible for $P(0)$!

(Within 1 bit of) any b-bit string possible!

(Within 1 bit of) b-bits are missing: one $P(i)$.

Within 1 of optimal number of bits.

Runtime.

Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $n-1$ polynomial $n+k$ times using $\log p$-bit numbers. $O(kn\log^2 p)$.
2. Reconstruct secret by solving system of n equations using $\log p$-bit arithmetic. $O(n^3\log^2 p)$.
3. Matrix has special form so $O(n\log n\log^2 p)$ reconstruction.

Faster versions in practice are almost as efficient.

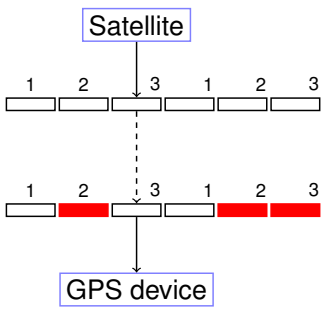
A bit of counting.

What is the number of degree d polynomials over $GF(m)$?

- m^{d+1}: $d+1$ coefficients from $\{0,\ldots,m-1\}$.
- m^{d+1}: $d+1$ points with y-values from $\{0,\ldots,m-1\}$

Infinite number for reals, rationals, complex numbers!

Erasure Codes.



3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1,1,and 3.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

On Friday!