CS70: Jean Walrand: Lecture 23.

What do we learn from observations?

1. Probability Basics Review
2. Examples
3. Conditional Probability

Probability Basics Review

Setup:

- Random Experiment.
  Flip a coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, Ω.
    $\Omega = \{HH, HT, TH, TT\}$
  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \leq Pr[\omega] \leq 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$.
  - **Event:** $A \subseteq \Omega, Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$.
    $Pr[\text{at least one H out of two tosses}] = Pr[HT, TH, HH] = 3/4$

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

Calculation.

Stirling formula (for large n):

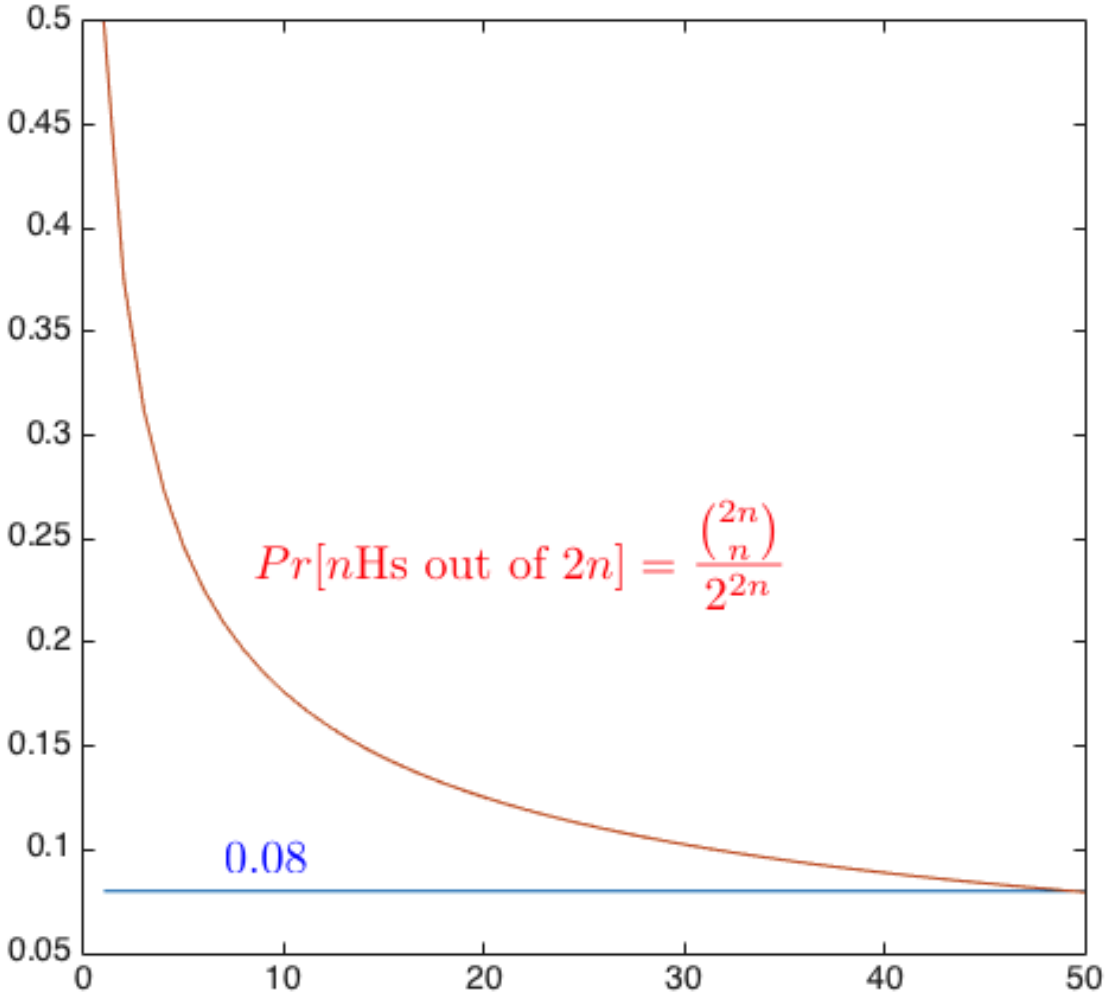
$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] \approx \frac{\frac{4^{50}}{\sqrt{50\pi}}}{2^{100}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

```
irb(main):006:0> (50*3.1415926)**(0.5)
=> 12.5331412662588
irb(main):007:0> 1/12.5331412662588
=> 0.0797884567608089
```

Exactly 50 heads in 100 coin tosses.

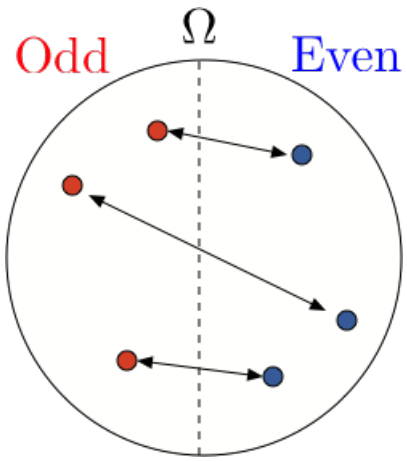


Probability of even number of heads in 57 coin tosses

Ω = 57 coin tosses. $|\Omega| = 2^{57}$.

Let $E = \{\omega \in \Omega \mid$ number of Hs in ω is even$\}$.

Fact: $P(E) = 1/2$.



Proof:
Consider the correspondence:

$$HA_1A_2\cdots A_{56} \leftrightarrow TA_1A_2\cdots A_{56}$$

Here, $A_1, \ldots, A_{56}$ are 56 coin flips.

It matches every even sequence to an odd sequence, and conversely.

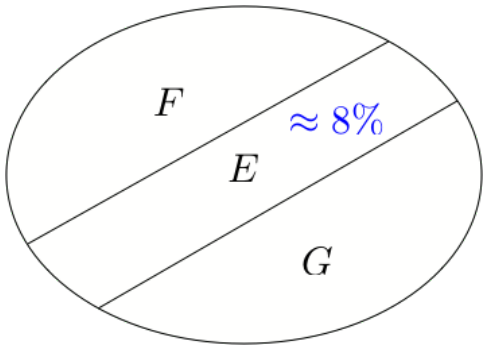
Hence, there are exactly as many odd as even sequences.

$$\Rightarrow |E| = |\Omega|/2 \Rightarrow P(E) = \frac{|E|}{|\Omega|} = \frac{1}{2}$$

Probability more heads than tails in 100 coin tosses.

Ω = 100 coin tosses.
$|\Omega| = 2^{100}$.

Recall event E = 'equal heads and tails'
Event F = 'more heads than tails'
Event G = 'more tails than heads'
A 1-to-1 correspondence between outcomes in F and G!
$|F| = |G|$.
E, F and G are disjoint.
$Pr[E] \approx 8\%$.



$|\Omega| = |E| + |F| + |G|$.
$\Rightarrow 1 = Pr[\Omega] = Pr[E] + 2Pr[F] \approx 8\% + 2Pr[F]$.

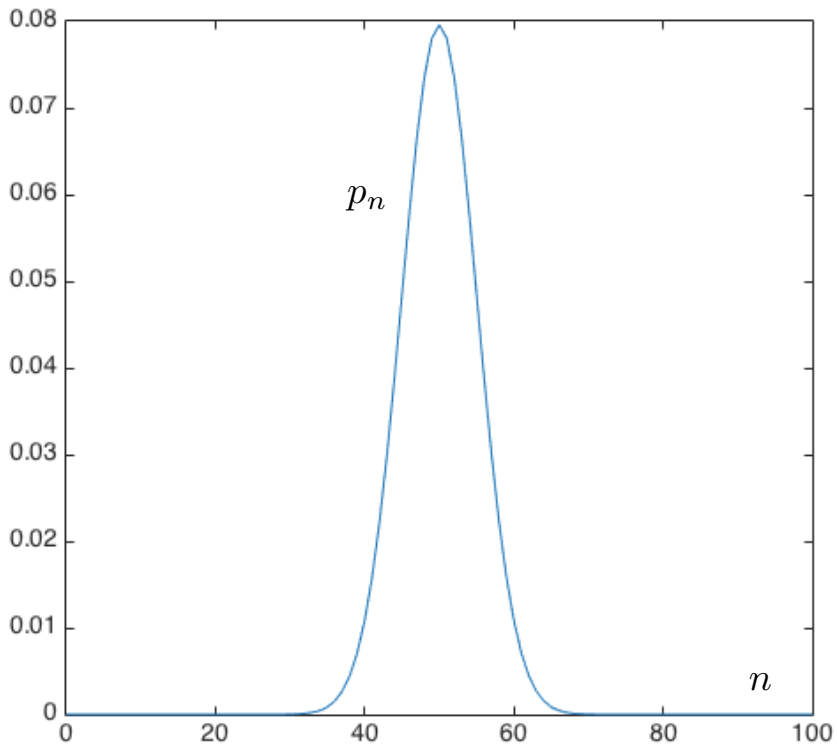
Solve for $|F|$:

$$|F| \approx 46\%$$

Probability of n heads in 100 coin tosses.

Ω = 100 coin tosses.
$|\Omega| = 2^{100}$.

Event E_n = 'n heads'; $|E_n| = \binom{100}{n}$



$$p_n = Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$$

Roll a red and a blue die.

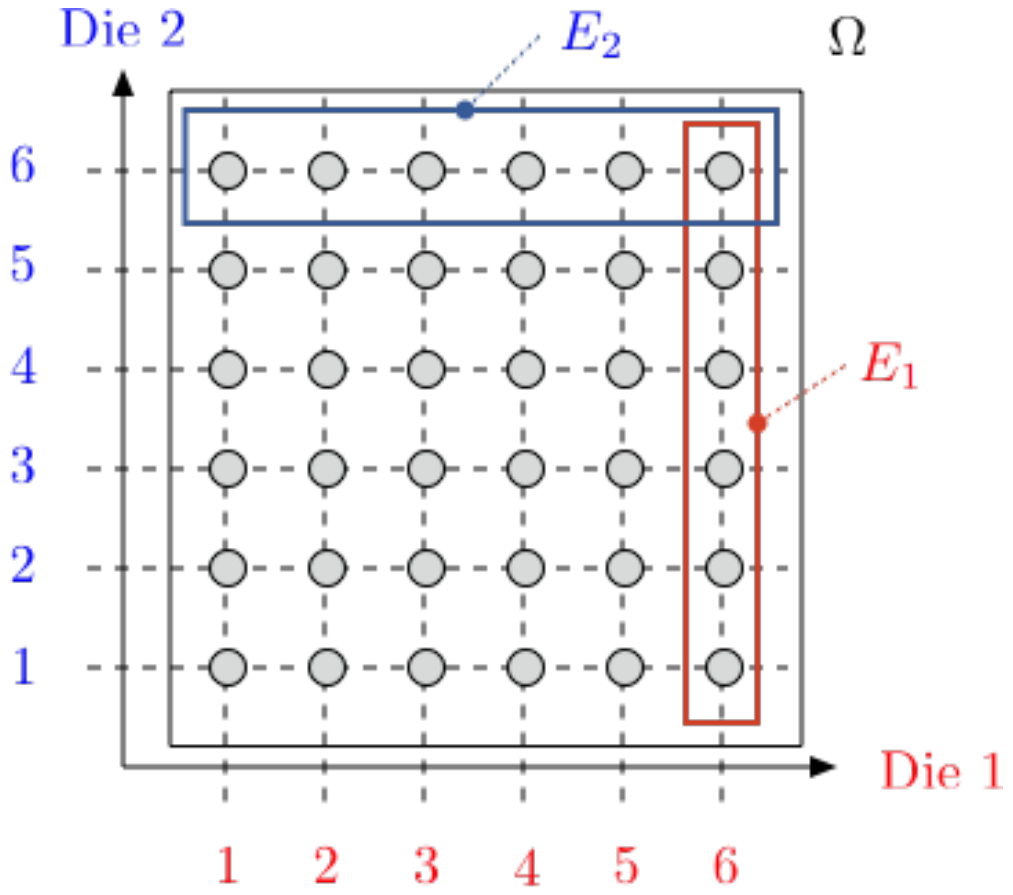
$$\Omega = \{(a,b) : 1 \leq a,b \leq 6\} = \{1,2,\ldots,6\}^2.$$

Uniform: $Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36}$ for all ω.

What is the probability of

1. the red die showing 6?
   $E_1 = \{(6,b) : 1 \leq b \leq 6\}$, $|E_1| = 6$, $Pr[E_1] = |E_1|/|\Omega| = 1/6$
2. at least one die showing 6?
   $E_1 = \{(6,b) : 1 \leq b \leq 6\}$ = red die shows 6
   $E_2 = \{(a,6) : 1 \leq a \leq 6\}$ = blue die shows 6
   $E = E_1 \cup E_2$ = red or blue die (or both) show 6
   $|E| = |E_1| + |E_2| - |E_1 \cap E_2|$ [Inclusion/Exclusion]
   $|E_1 \cap E_2| = \{(6,6)\}$
   $|E| = 6 + 6 - 1 = 11$
   $Pr[E] = 11/36$

Roll a red and a blue die.



$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$E_1 =$ 'Red die shows 6'; $E_2 =$ 'Blue die shows 6'

$E_1 \cup E_2 =$ 'At least one die shows 6'

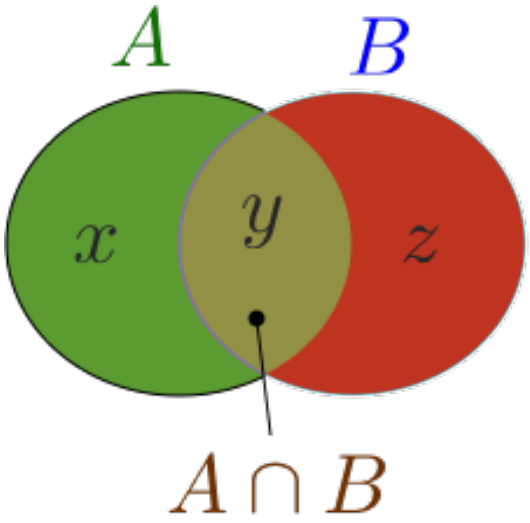
$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Inclusion/Exclusion

Note that,

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

whether or not the sample space has uniform distribution.



$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

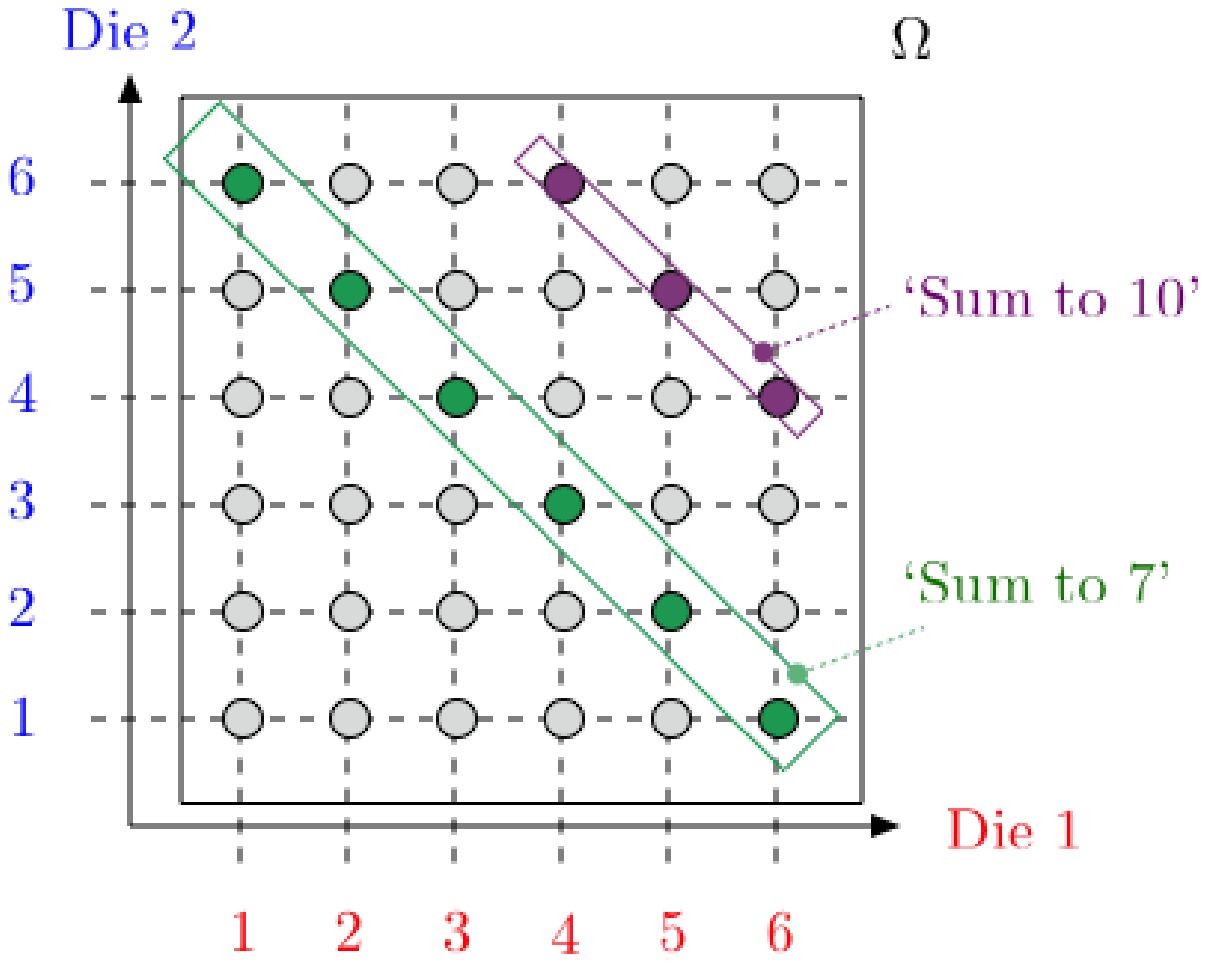
Roll a red and a blue die.

$$\Omega = \{(a,b) : 1 \leq a,b \leq 6\} = \{1,2,\ldots,6\}^2.$$

Uniform: $Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36}$ for all ω.
What is the probability of

1. the dice sum to 7?
   $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}; |E| = 6.$
   Counting argument:
   for each choice a of the value of the red die,
   there is exactly one choice $b = 7 - a$ for the blue die,
   so there are 6 total choices.
   $Pr[E] = |E|/|\Omega| = 6/36 = 1/6.$
2. the dice sum to 10?
   $E = \{(4,6),(5,5),(6,4)\}$
   $Pr[E] = |E|/|\Omega| = 3/36 = 1/12.$

Roll a red and a blue die.



$$Pr[\text{Sum to } 7] = \frac{6}{36} \qquad Pr[\text{Sum to } 10] = \frac{3}{36}$$

Roll two blue dice.

The key idea is that we do not distinguish the dice.

Roll die 1, then die 2. Then forget the order.

For instance, we consider that $(2,5)$ and $(5,2)$ are the same outcome.

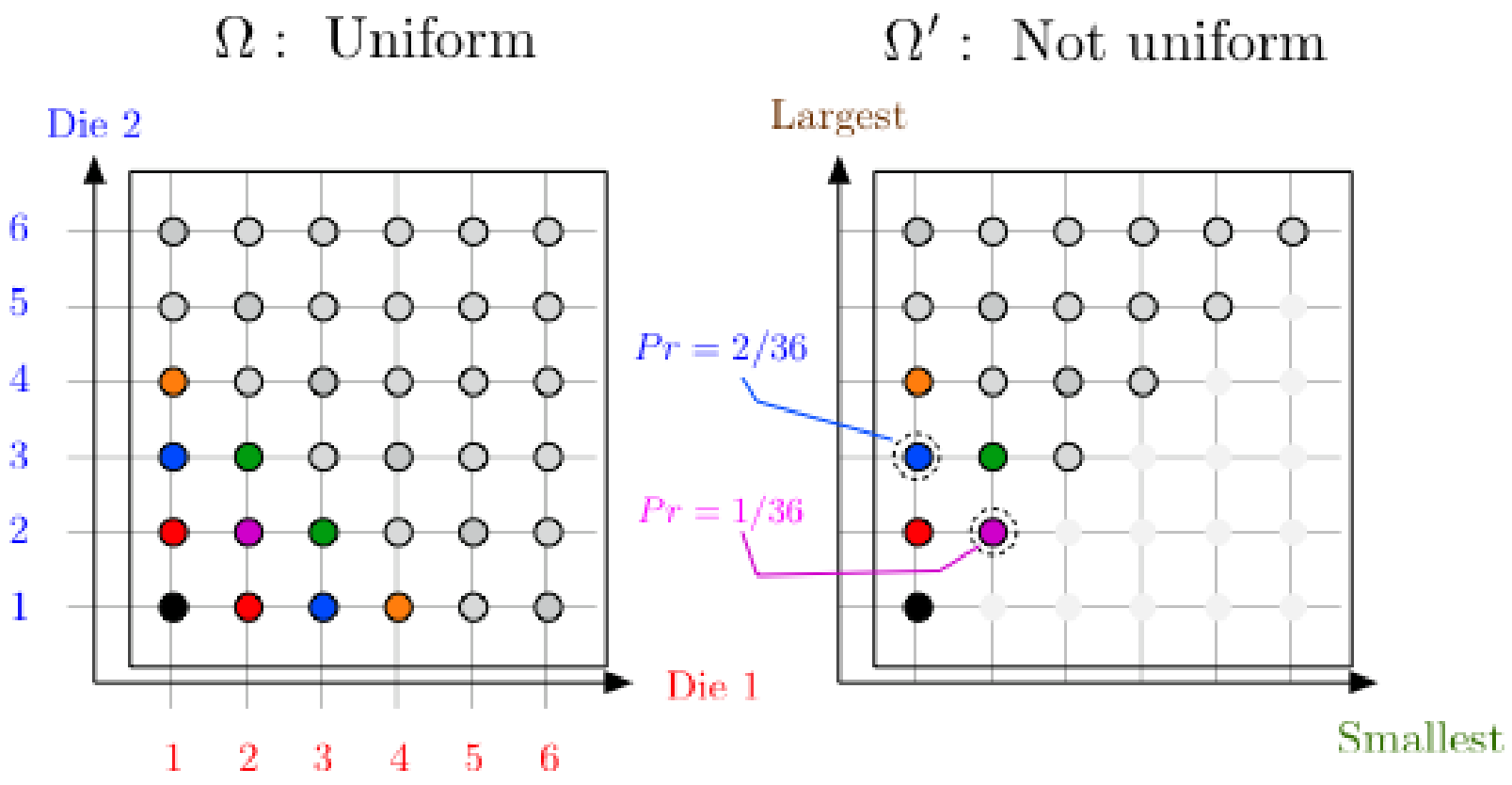
We designate this outcome by $(2,5)$.

Thus,

$$\Omega' = \{(a,b) \mid 1 \leq a \leq b \leq 6\}.$$

We see that $Pr[(1,3)] = \frac{2}{36}$ and $Pr[(2,2)] = \frac{1}{36}$.

Roll two blue dice

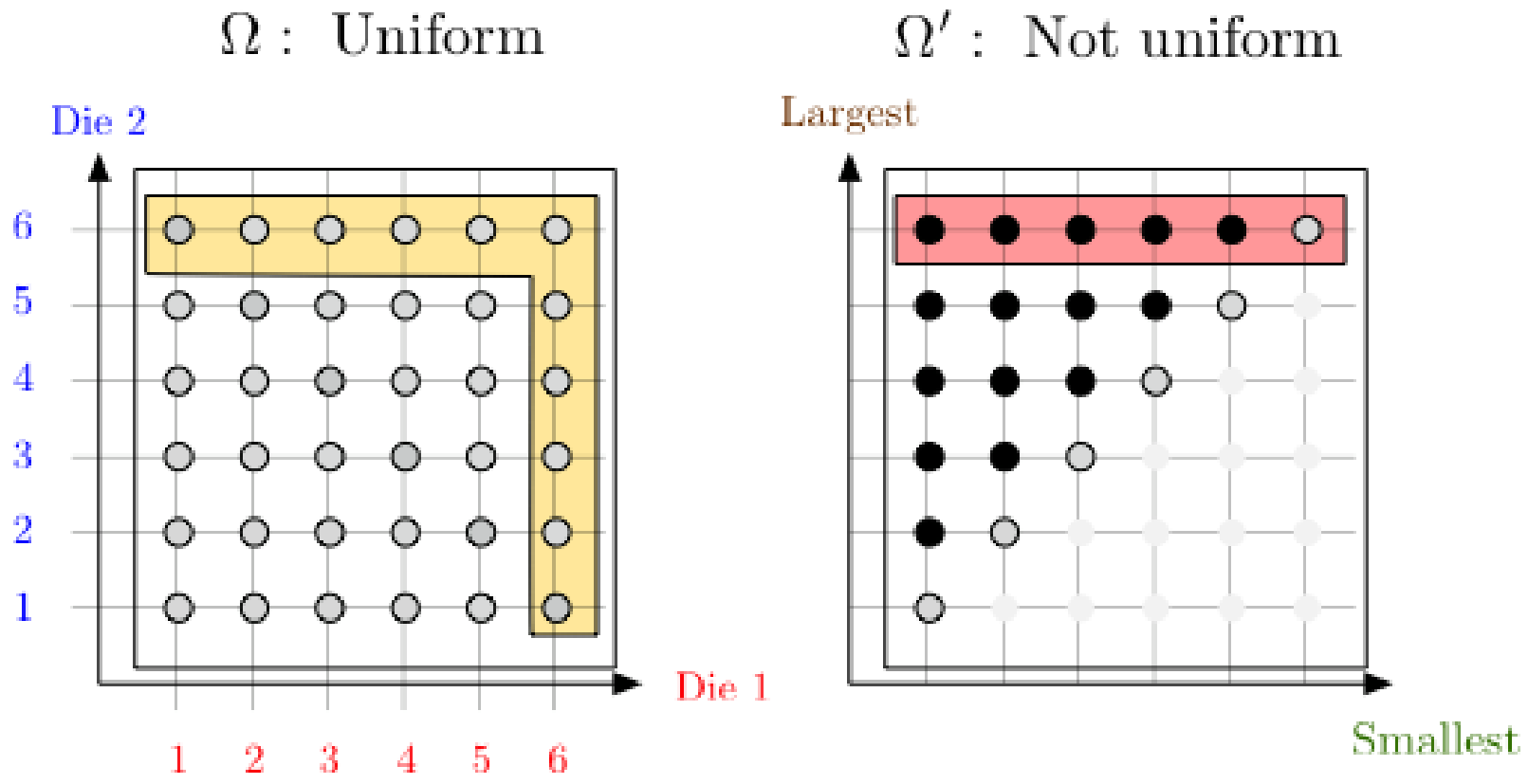


Two different models of the same random experiment.

$$\text{In } \Omega', Pr[(1,3)] = \frac{2}{36} \text{ and } Pr[(2,2)] = \frac{1}{36}$$

Roll two blue dice.

Now what is the probability of at least one die showing 6?



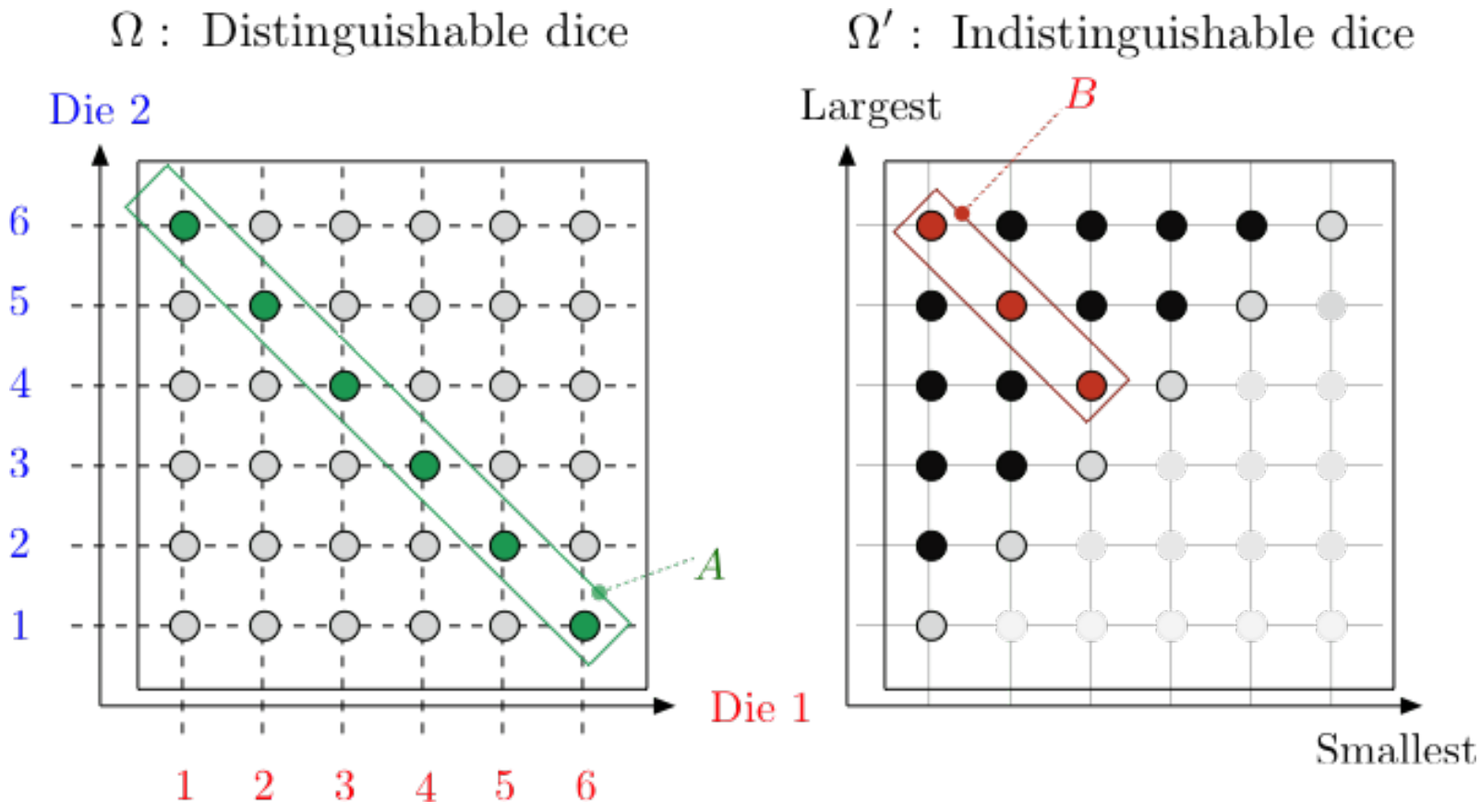
In Ω, $Pr[A] = \frac{11}{36}$; in Ω', $Pr[B] = 5 \times \frac{2}{36} + 1 \times \frac{1}{36}$.

Of course, this is the same as for distinguishable dice!

The event does not depend on the dice being distinguishable.

Roll two blue dice.

What is the probability of the dice sum to 7?



In $\Omega, Pr[A] = \frac{6}{36}$; in $\Omega', Pr[B] = 3 \times \frac{2}{36}$.

Of course, this is the same as for distinguishable dice!

The event does not depend on the dice being distinguishable.

Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = 2/3$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = (\frac{2}{3})^3$; $Pr[HHT] = (\frac{2}{3})^2(\frac{1}{3})$; ...
- Toss a fair coin until you get a heads.
  - $\Omega = \{H, TH, TTH, TTTH, \ldots\}$
  - $Pr[H] = \frac{1}{2}, Pr[TH] = \frac{1}{4}, Pr[TTH] = \frac{1}{8}$
  - Still sums to 1. Indeed $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$.

Set notation review

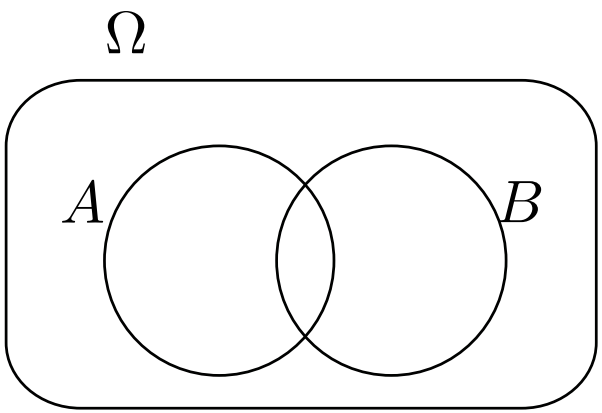


Figure : Two events

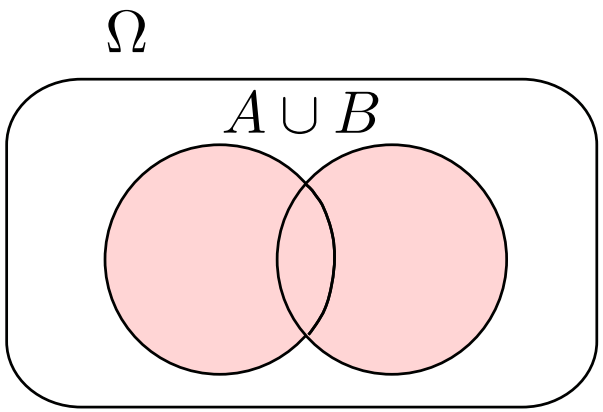


Figure : Union (or)

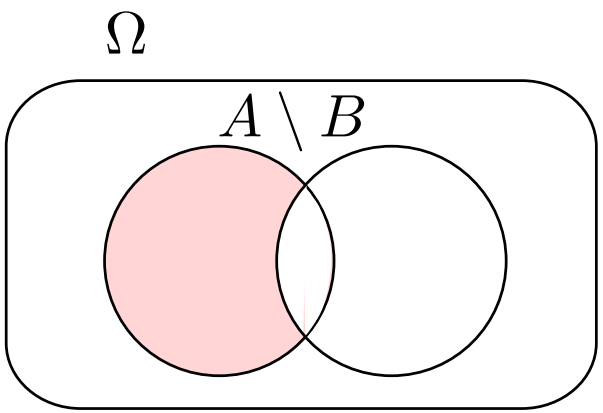


Figure : Difference (A, not B)

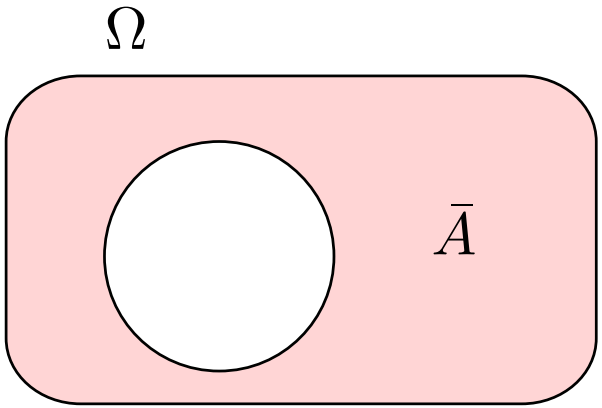


Figure : Complement (not)

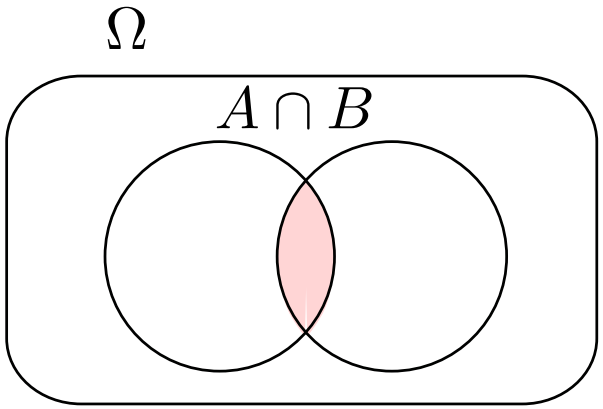


Figure : Intersection (and)

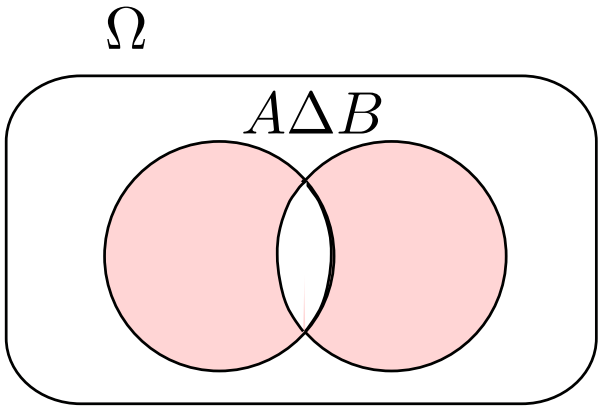
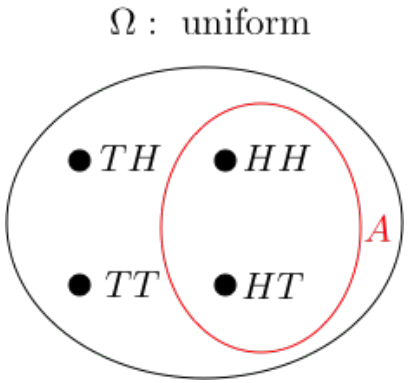


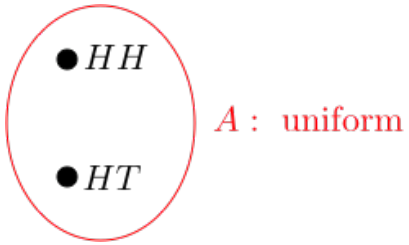
Figure : Symmetric difference (only one)

Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
$\Omega = \{HH, HT, TH, TT\}$; Uniform probability space.
Event $A =$ first flip is heads: $A = \{HH, HT\}$.



New sample space: A; uniform still.



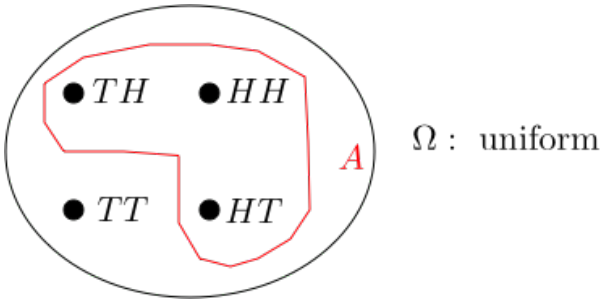
Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B **given** A is $1/2$.

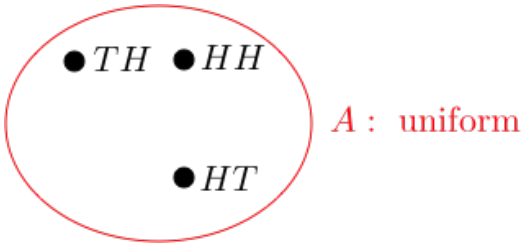
A similar example.

Two coin flips. One of the flips is heads.
Probability of two heads?

$\Omega = \{HH, HT, TH, TT\}$; uniform.
Event $A =$ one flip is heads. $A = \{HH, HT, TH\}$.



New sample space: A; uniform still.

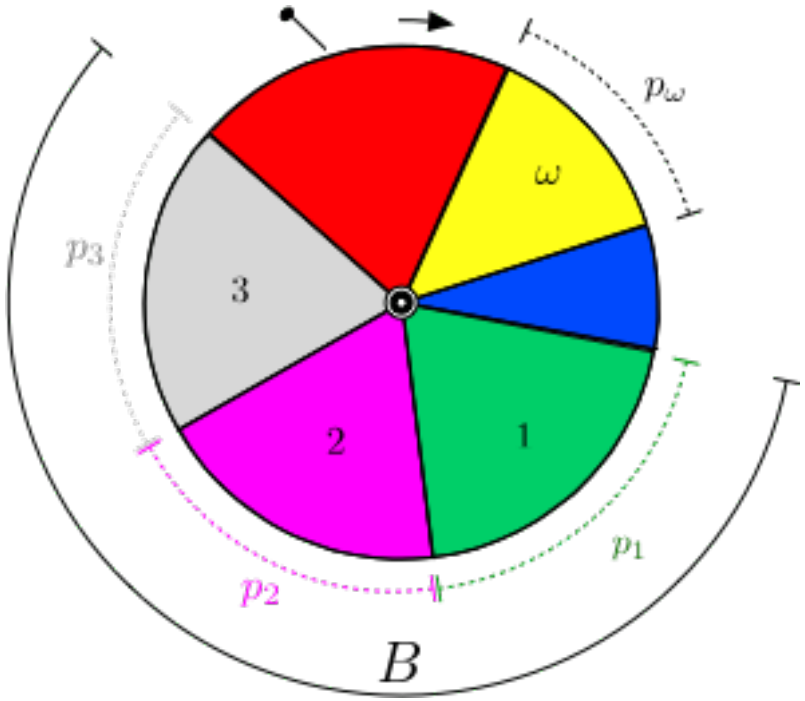


Event $B =$ two heads.

The probability of two heads if at least one flip is heads.
The probability of B given A is $1/3$.

Conditional Probability: A non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

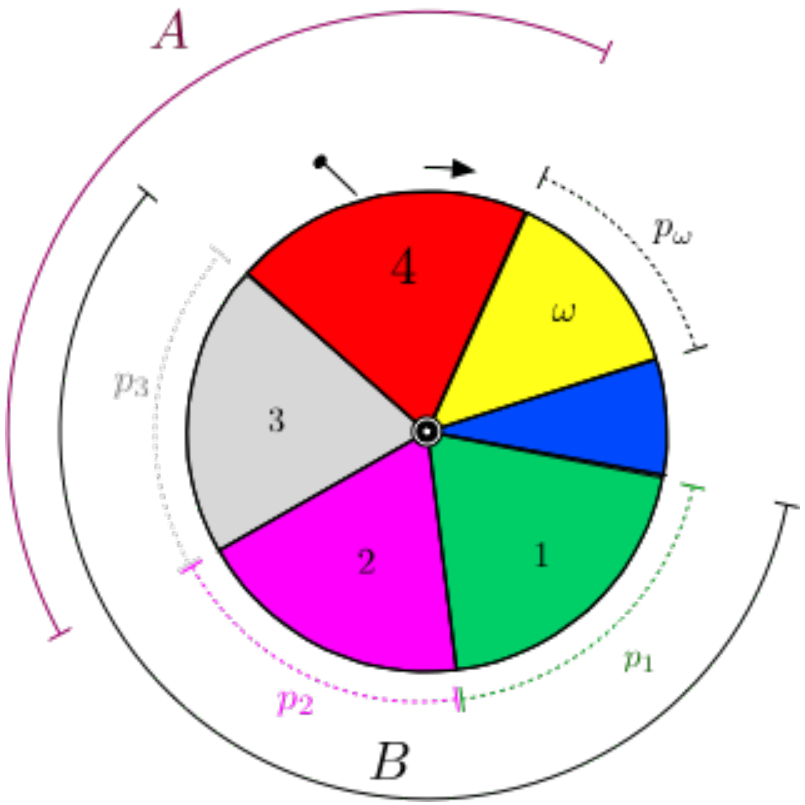


$$Pr[3|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{p_3}{Pr[B]}.$$

$$\omega \notin B \Rightarrow Pr[\omega|B] = 0.$$

Another non-uniform example

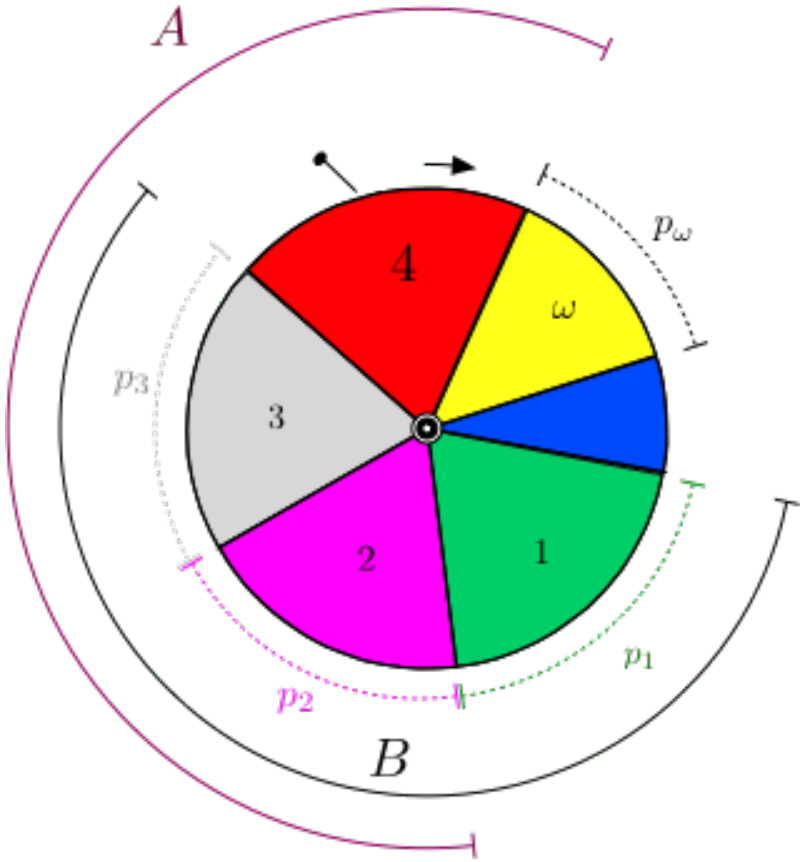
Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.



$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

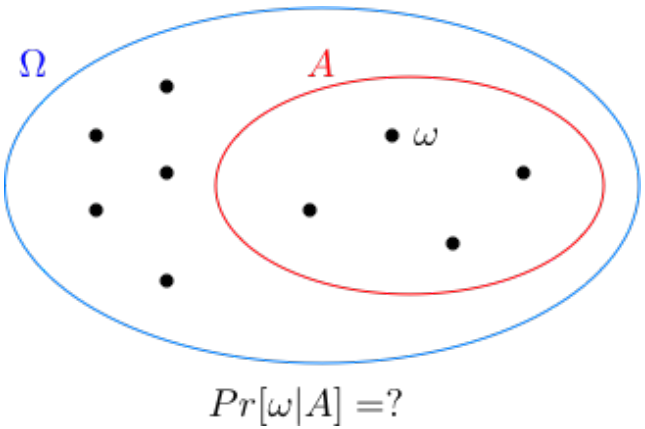
Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.



$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Conditional Probability.



If in A, what is the probability of outcome ω?

If $\omega \notin A$, probability is 0
Otherwise: Ratio of probability of ω to total probability of A

$$Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}$$

Uniform Probability Space:
Ratio of $1/|\Omega|$ to $|A|/|\Omega| \implies 1/|A|$. (Makes sense!)

Conditional Probability.

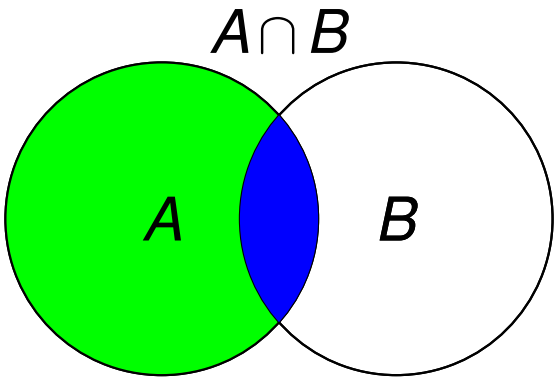
Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \sum_{\omega \in B} Pr[\omega|A] = \frac{\sum_{\omega \in A \cap B} Pr[\omega]}{Pr[A]} = \frac{Pr[A \cap B]}{Pr[A]}$$

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



In A!
In B?
Must be in $A \cap B$.

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

What do we learn from observations?

You observe that the event B occurs.

That changes your information about the probability of every event A.

The conditional probability of A given B is

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}.$$

Note: $Pr[A \cap B] = Pr[B] \times Pr[A|B] = Pr[A] \times Pr[B|A]$.