CS70: Jean Walrand: Lecture 25.

Causality, Independence, Collisions and Collecting

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1. Product Rule
2. Correlation and Causality
3. Independence
4. Balls in bins
5. Birthdays
6. Checksums
7. Collecting Coupons

Product Rule

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so that the result holds for $n+1$. $\square$

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- Smoking increases the probability of lung cancer by 17%.
- Smoking causes lung cancer.

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More about such questions later. For fun, check “N. Taleb: Fooled by randomness.”

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See next slide.

Example 1

Flip two fair coins. Let

- $A =$ 'first coin is H' $= \{HT, HH\}$;
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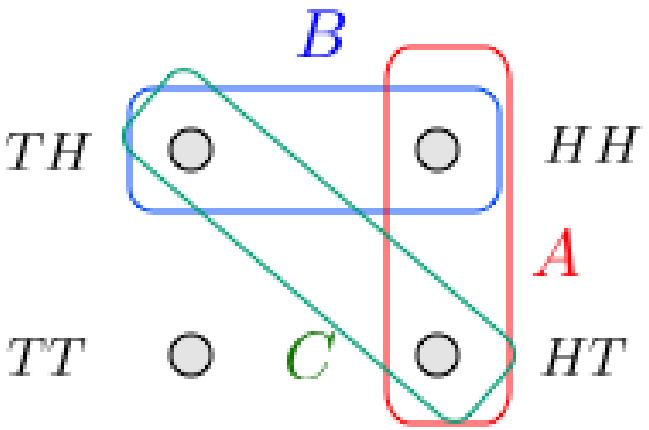

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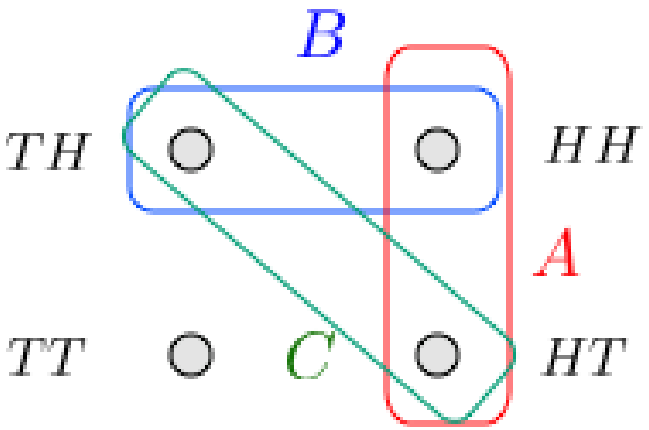

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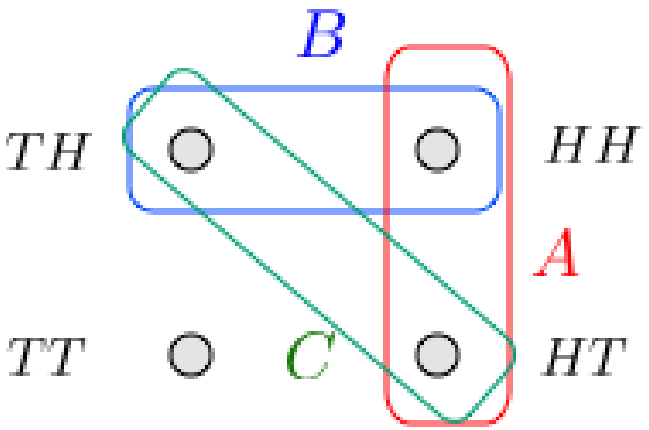

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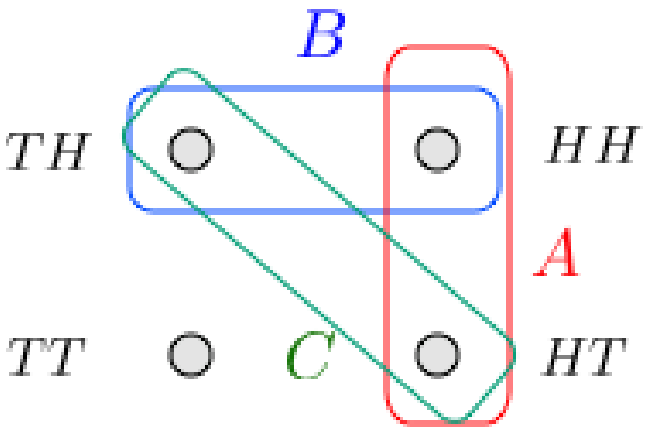

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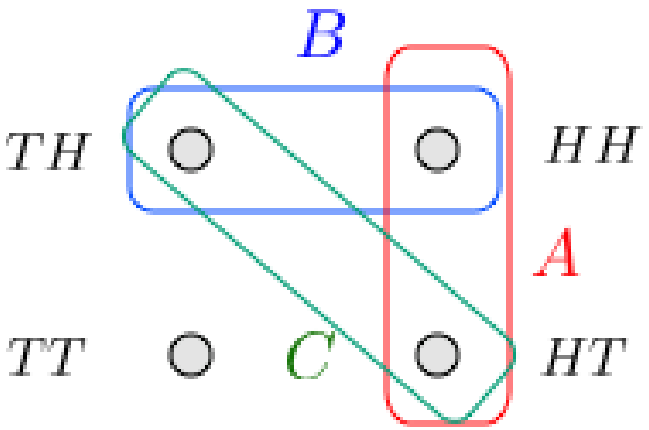

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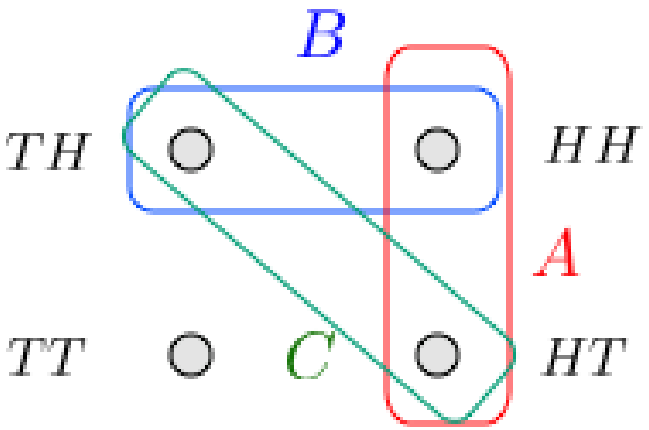

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$A \cap B, C$ are not independent. $(Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B]Pr[C].)$

If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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(a) If the events $\{A_j, j \in J\}$ are mutually independent and if K_1 and K_2 are disjoint finite subsets of J, then

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Proof:
See homework. $\square$

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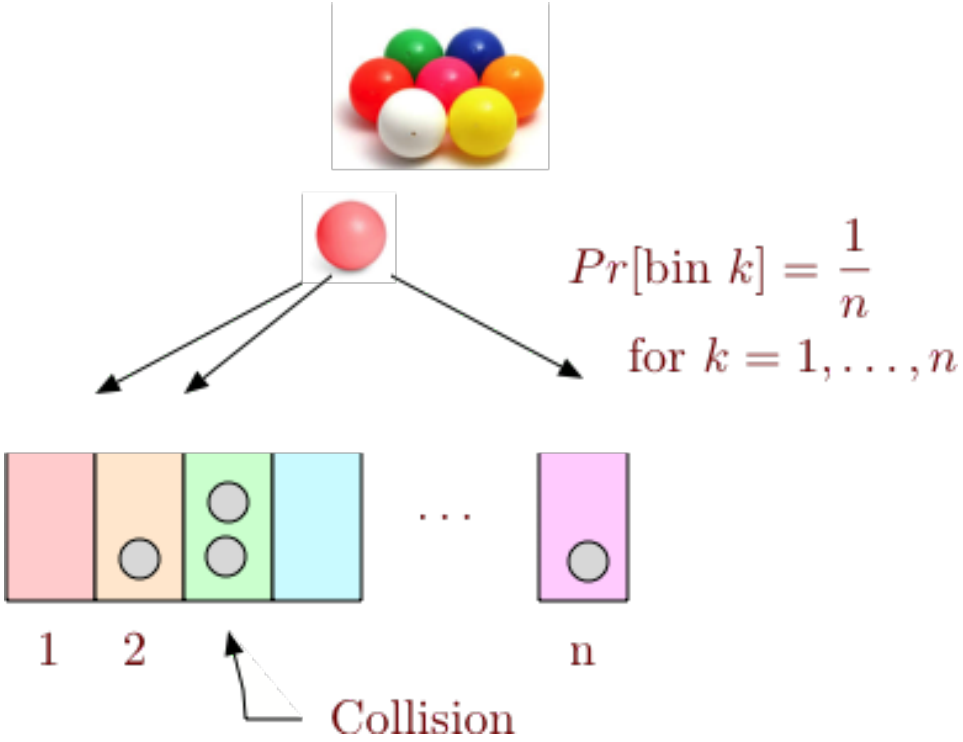

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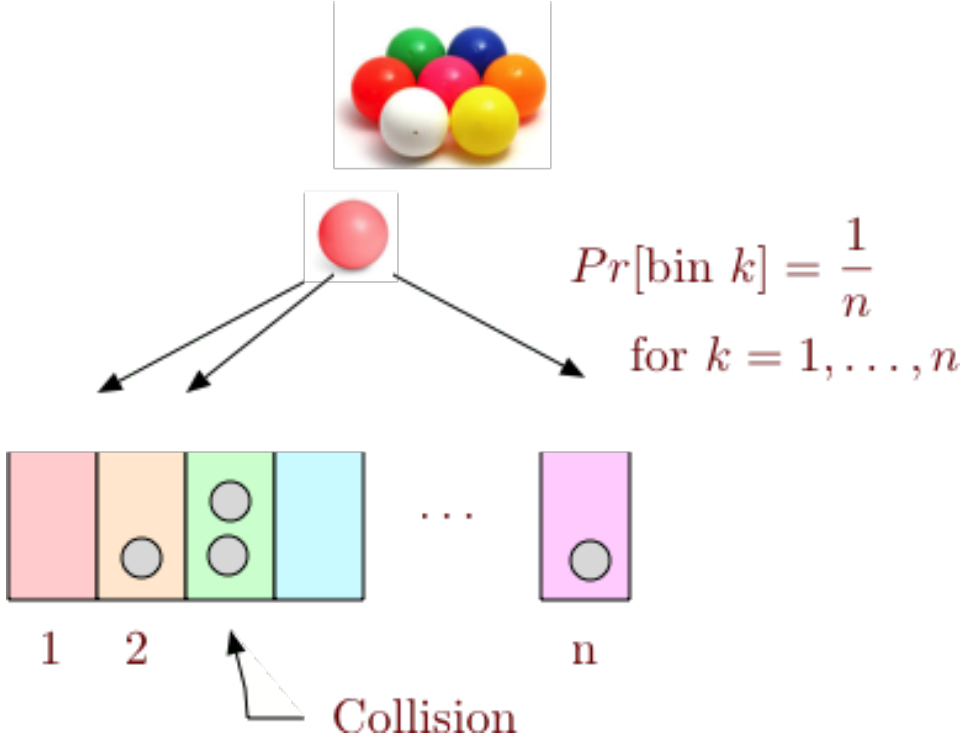

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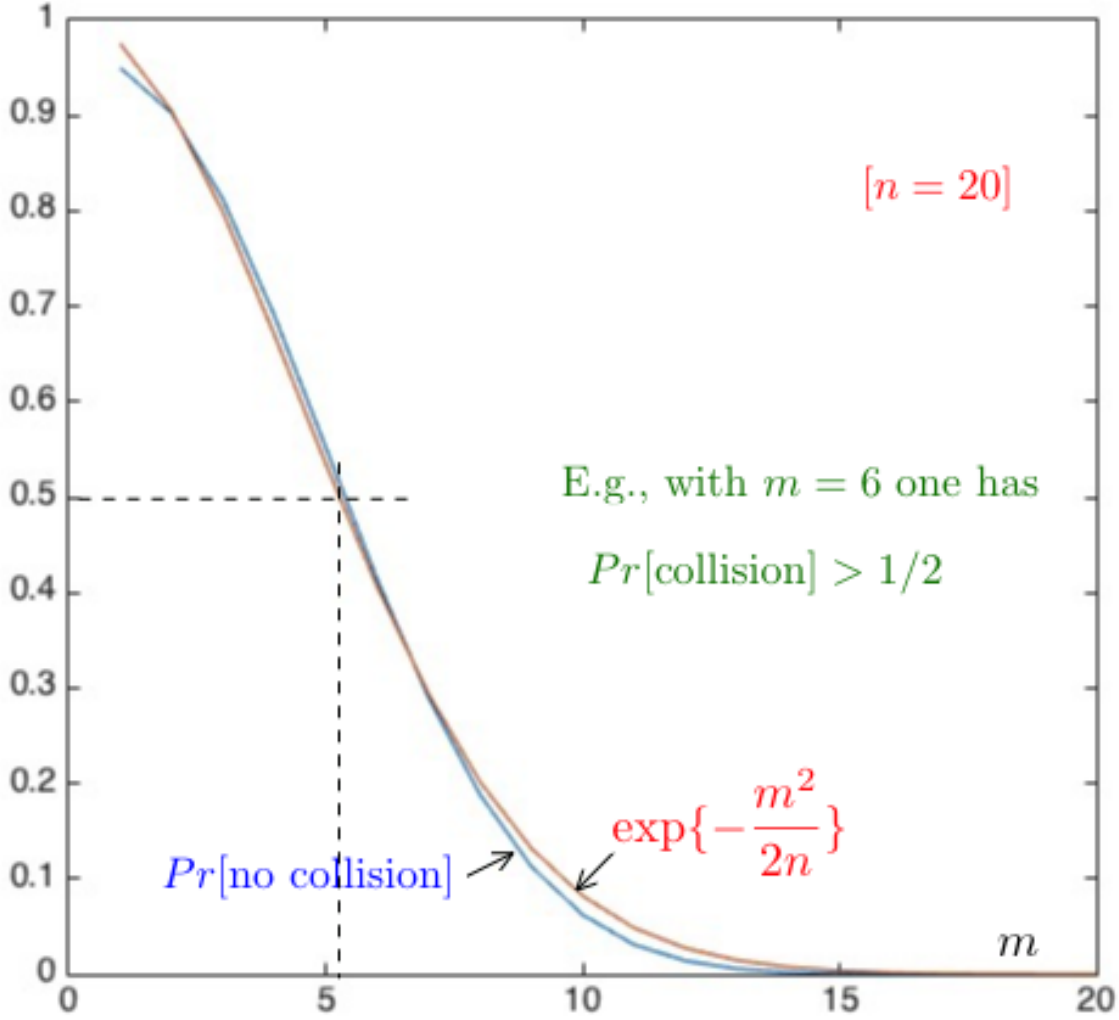

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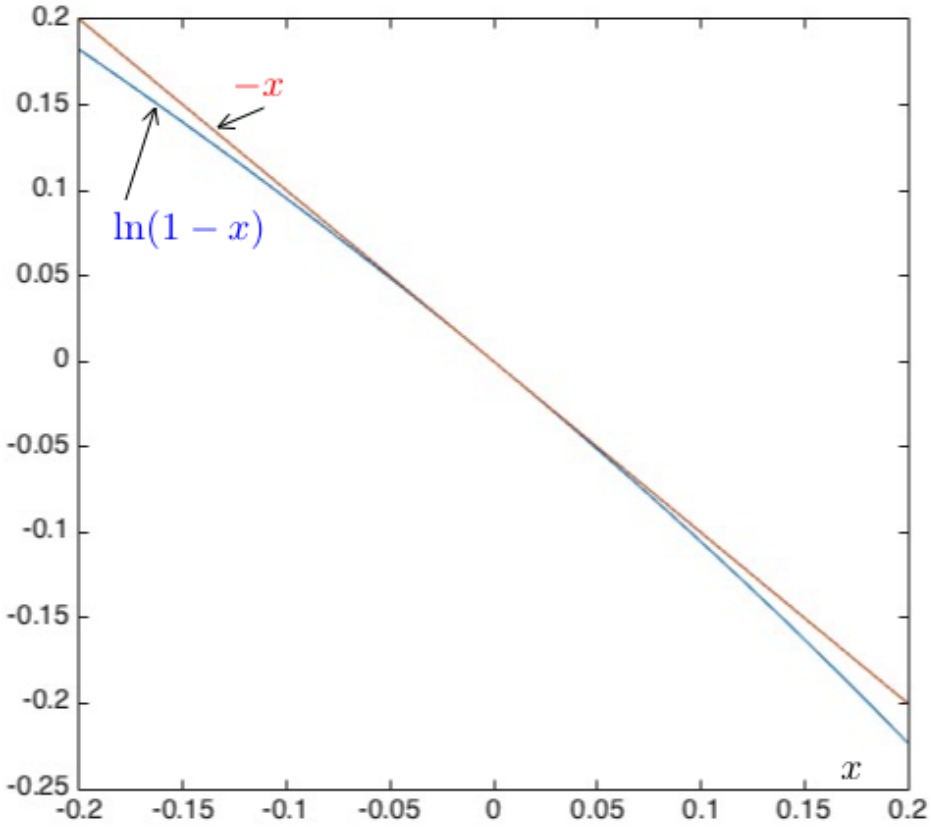
$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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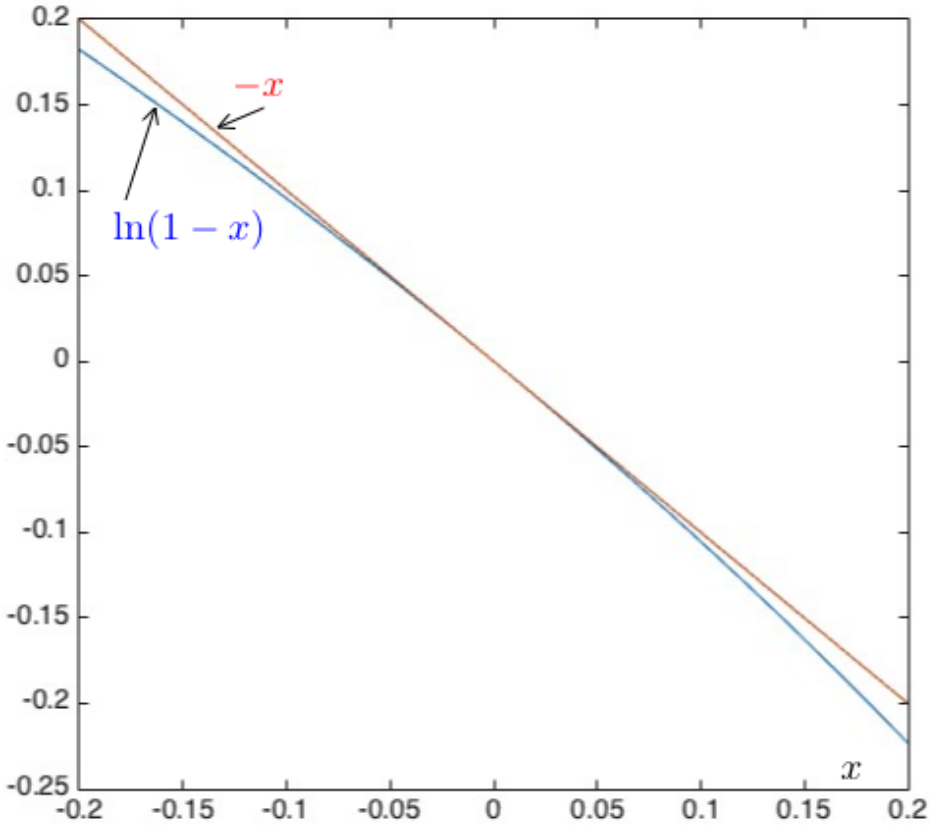

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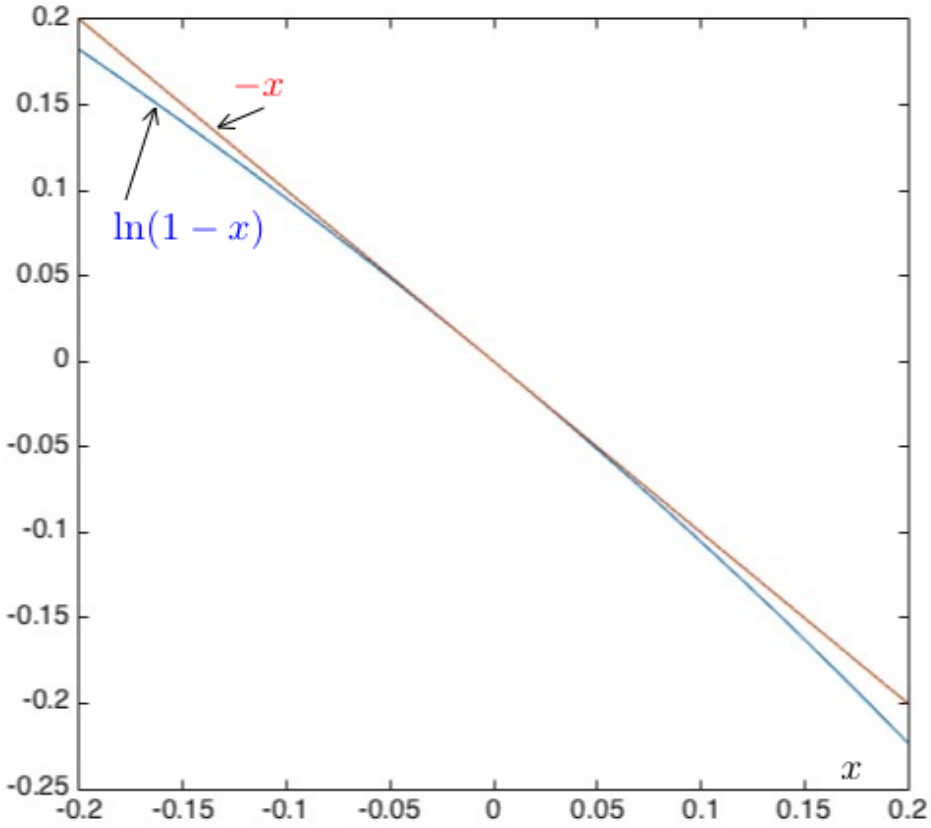

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

Sum of consecutive integers

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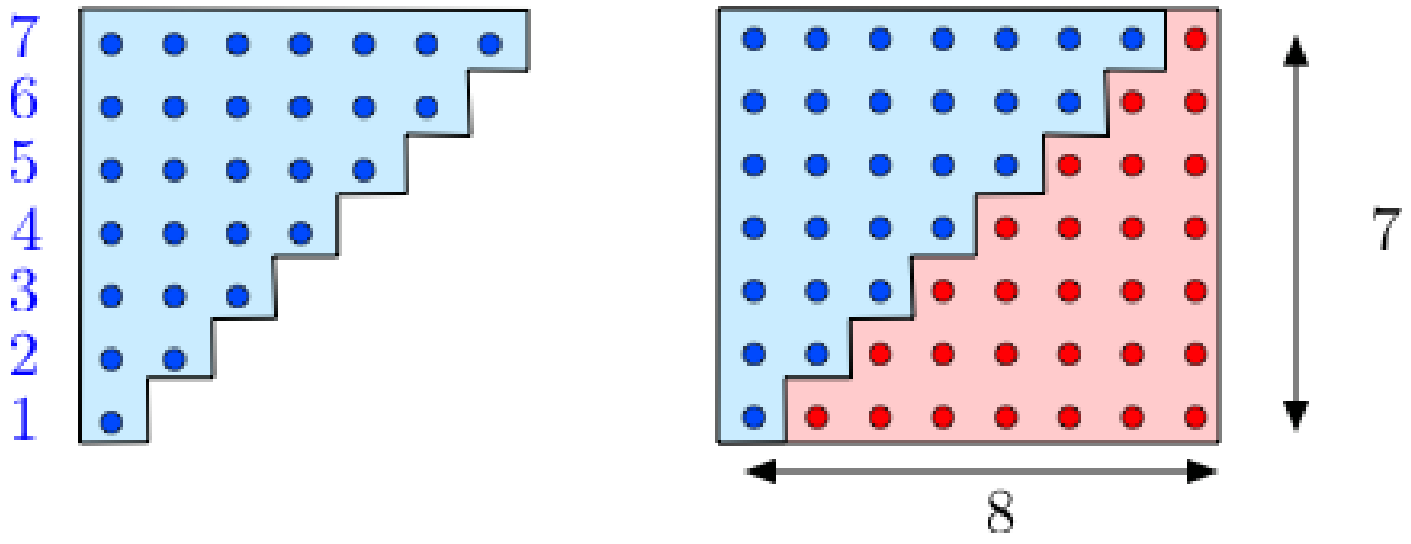
Recall this useful fact:



$$1 + 2 + 3 + \cdots + 7 = \frac{7 \times 8}{2}$$

$$1 + 2 + 3 + \cdots + n = \frac{n \times (n+1)}{2}$$

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$^{(*)}$ $\log_2(x) = \ln(x)/\ln(2)$. Indeed: $\ln(x) = \log_2(x)\ln(2)$ since $e^{\log_2(x)\ln(2)} = [e^{\ln(2)}]^{\log_2(x)}$

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$^{(*)}$ $\log_2(x) = \ln(x)/\ln(2)$. Indeed: $\ln(x) = \log_2(x)\ln(2)$ since $e^{\log_2(x)\ln(2)} = [e^{\ln(2)}]^{\log_2(x)} = 2^{\log_2(x)}$

Checksum

For b-bit checksums for m files, we claim that

$$Pr[\text{collision}] \leq \frac{1}{2^{10}} \text{ if } b \geq 2.9 \ln m + 9.$$

E.g., for $m = 10^{14}$ files, a 103-bit checksum suffices!

Derivation: b bits $\Leftrightarrow n = 2^b$ bins.

$$\begin{aligned}
&Pr[\text{collision}] \leq 2^{-10} \Leftrightarrow Pr[\text{no collision}] \geq 1 - 2^{-10} \\
&\Leftrightarrow \exp\{-\frac{m^2}{2n}\} \geq 1 - 2^{-10} \\
&\Leftrightarrow 1 - \frac{m^2}{2n} \geq 1 - 2^{-10} \Leftrightarrow \frac{m^2}{2n} \leq 2^{-10} \\
&\Leftrightarrow 2n \geq m^2 2^{10} \Leftrightarrow n \geq m^2 2^9 \Leftrightarrow 2^b \geq m^2 2^9 \\
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CS161

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n different baseball cards.

(Brian Wilson, Jackie Robinson, Roger Hornsby, ...)

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

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Ask $n\ln(2n)$ servers to get file with probability $\frac{1}{2}$.

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Use Reed-Solomon codes (or Tornado codes[1]).

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E.g., for $n=100$, around a factor of 4 better!

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