CS70: Lecture 3. Induction!

1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.
5. Two coloring map
6. Strengthening induction.
7. Horses with one color...
8. Try this at home.

The naturals.

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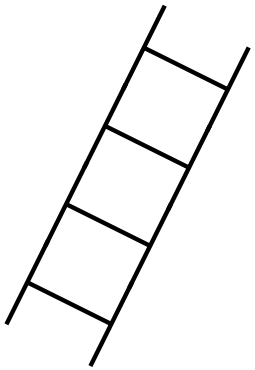
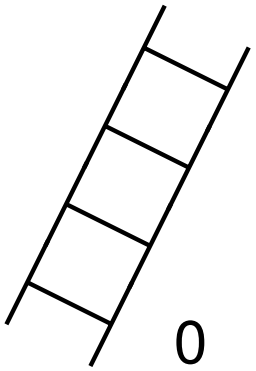


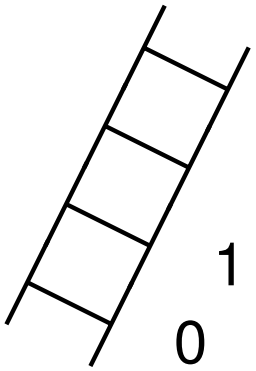
The naturals.

0,



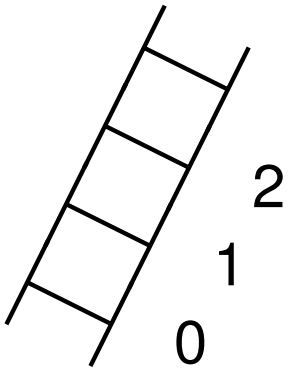
The naturals.

0, 1,



The naturals.

0, 1, 2,



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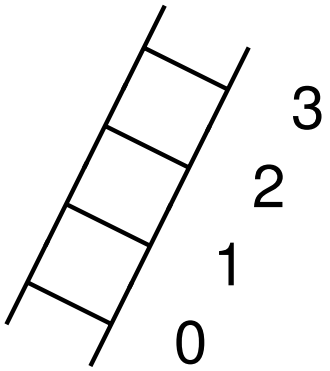
0, 1, 2, 3,



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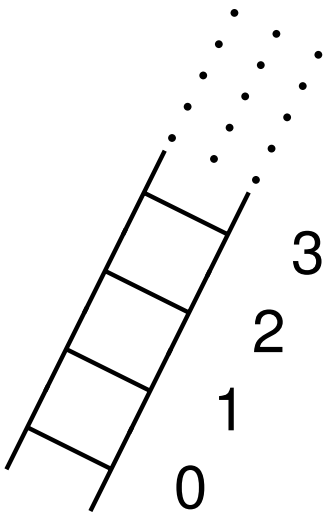

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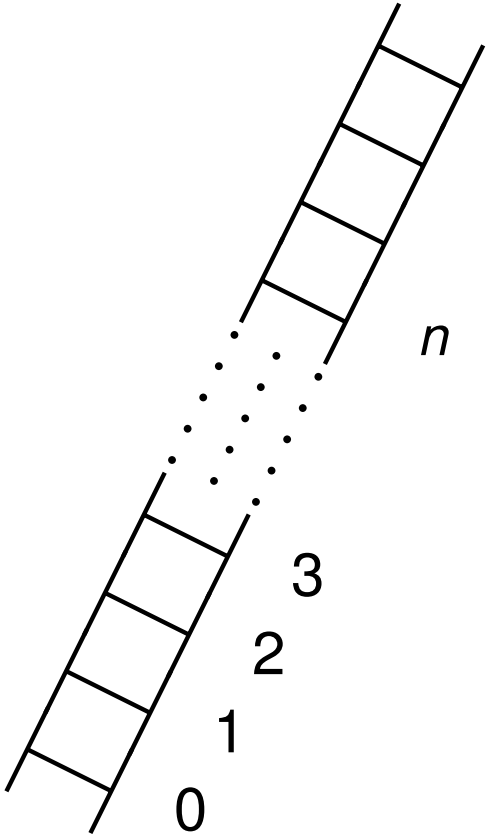

0, 1, 2, 3,

$\ldots, n,$

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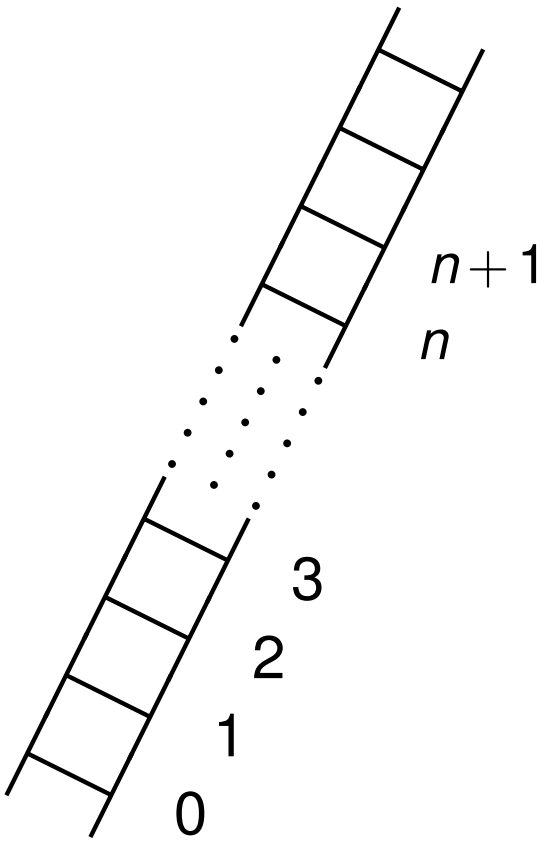
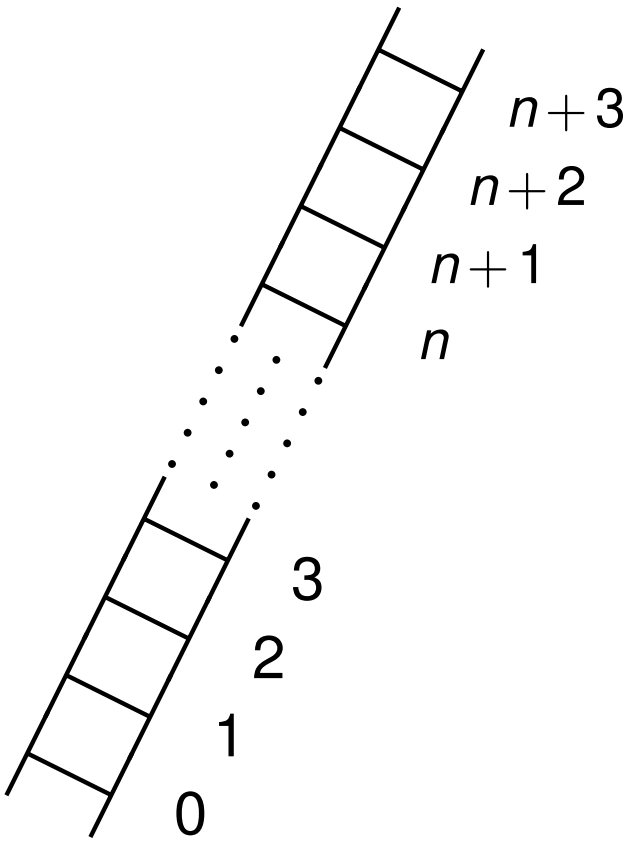


$0, 1, 2, 3,$

$\ldots, n, n+1,$

The naturals.



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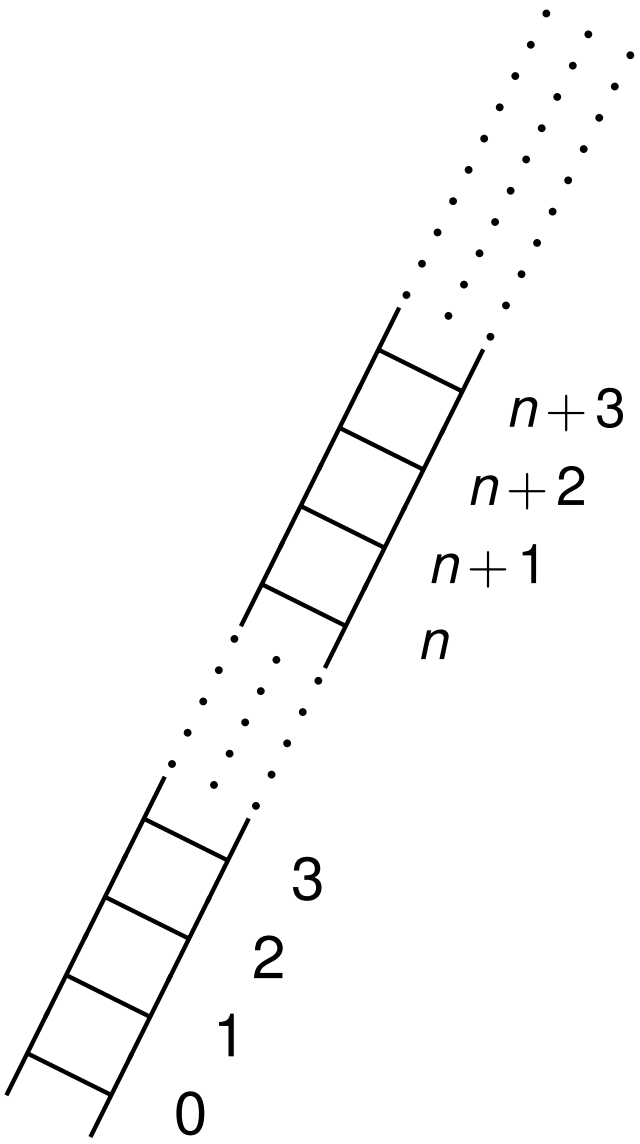

0, 1, 2, 3,

$\ldots, n, n+1, n+2, n+3, \ldots$

A formula.

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Teacher: Hello class.

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Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

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Teacher: Please add the numbers from 1 to 100.

Gauss: It's

A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$

A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

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Is predicate true for $n = k + 1$?

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1)$

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How about $k+2$.

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Predicate True for all natural numbers!

Proof by Induction.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

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The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

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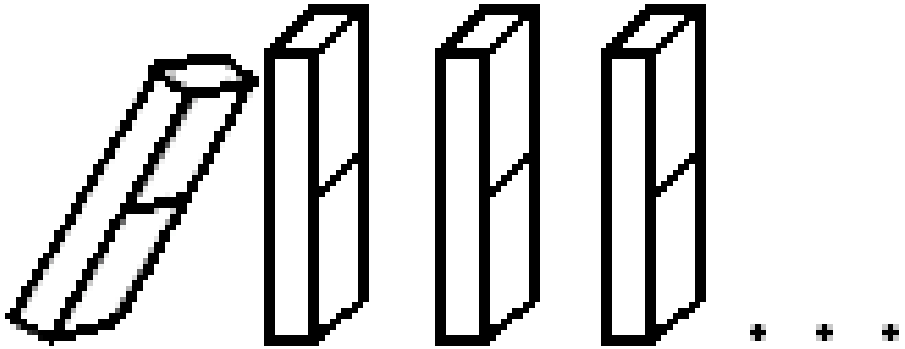
Notes visualization

An visualization: an infinite sequence of dominos.



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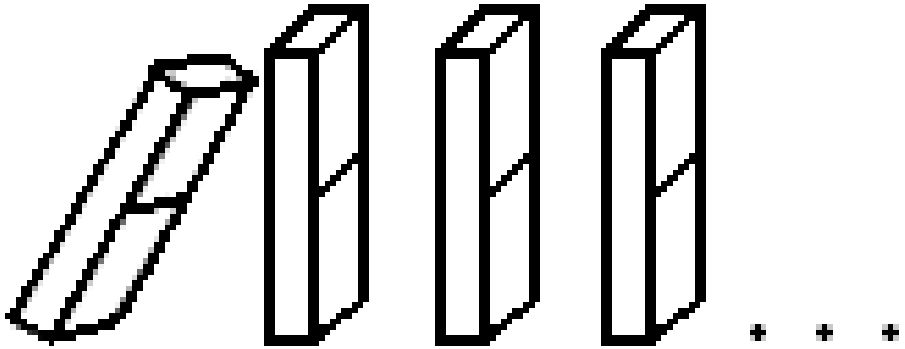

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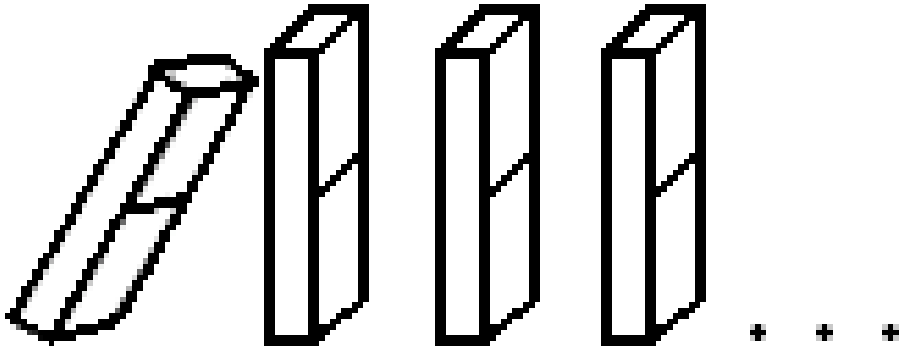

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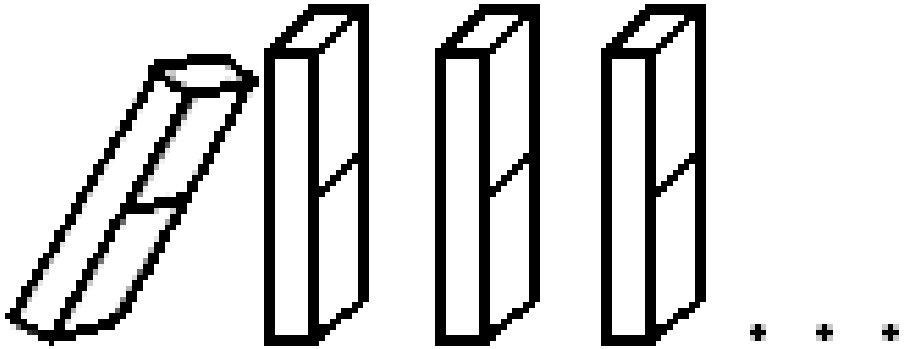

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Climb an infinite ladder?

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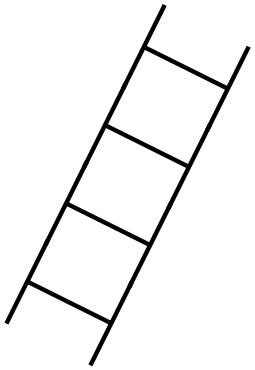

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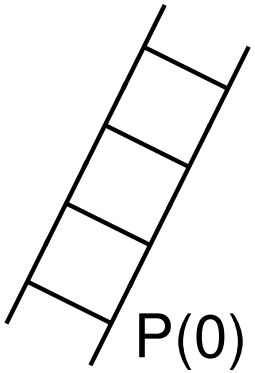

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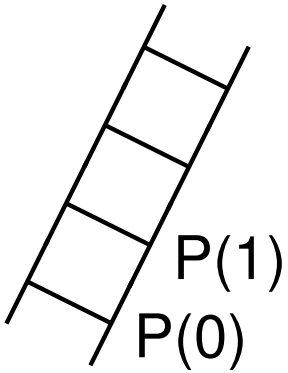

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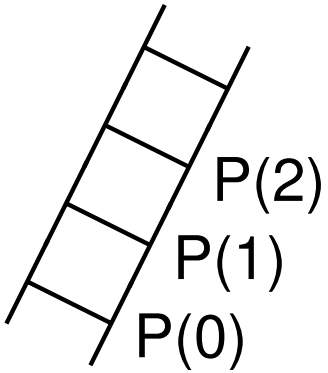

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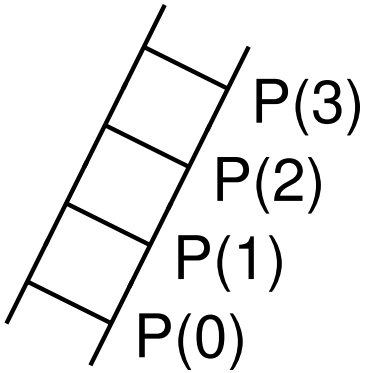

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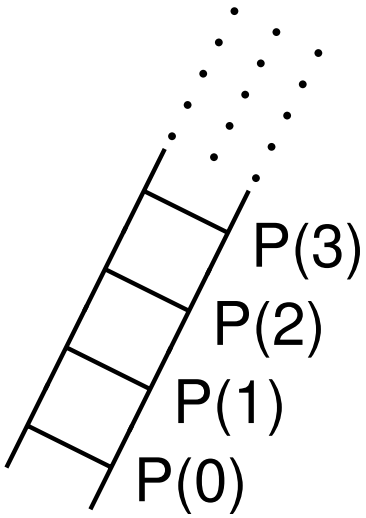

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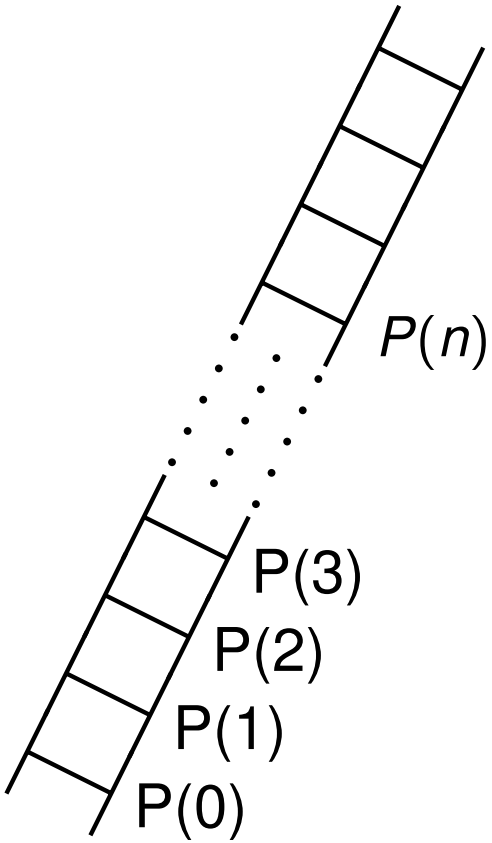

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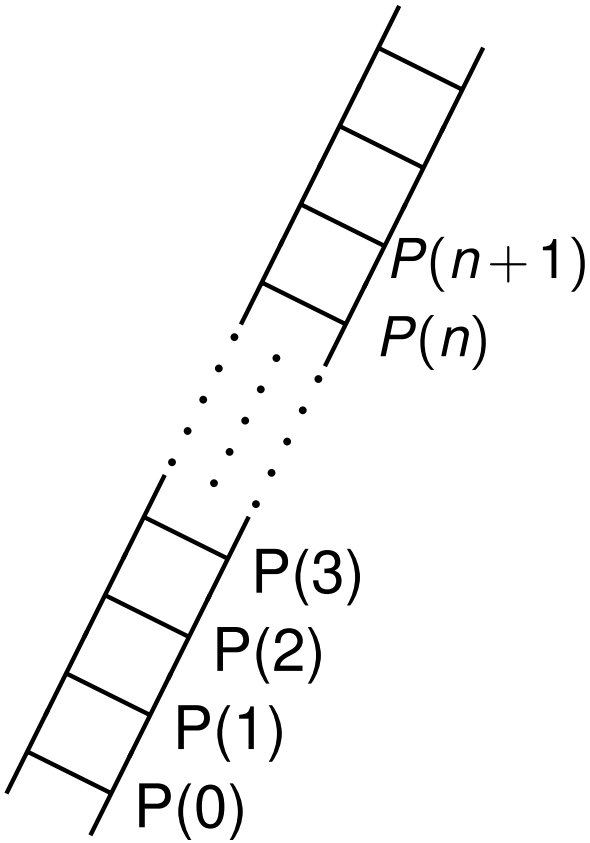

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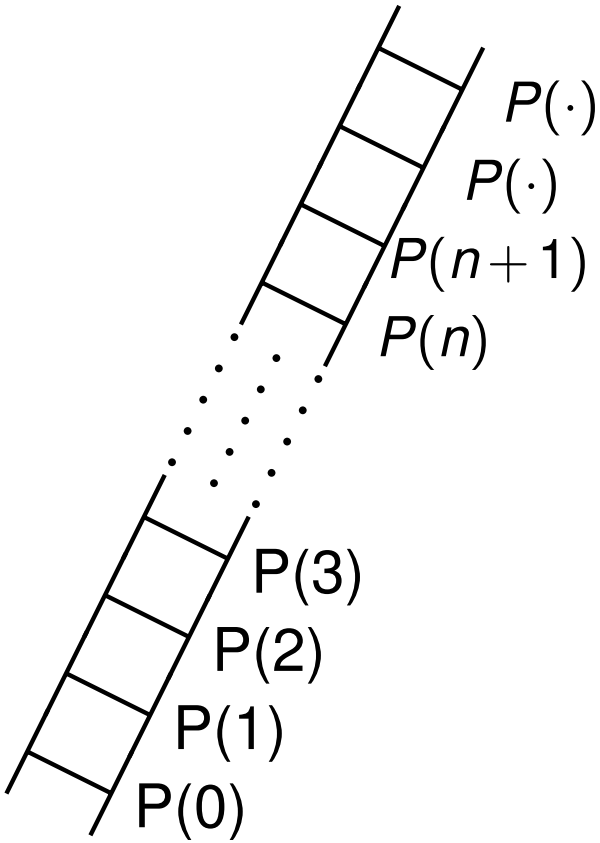

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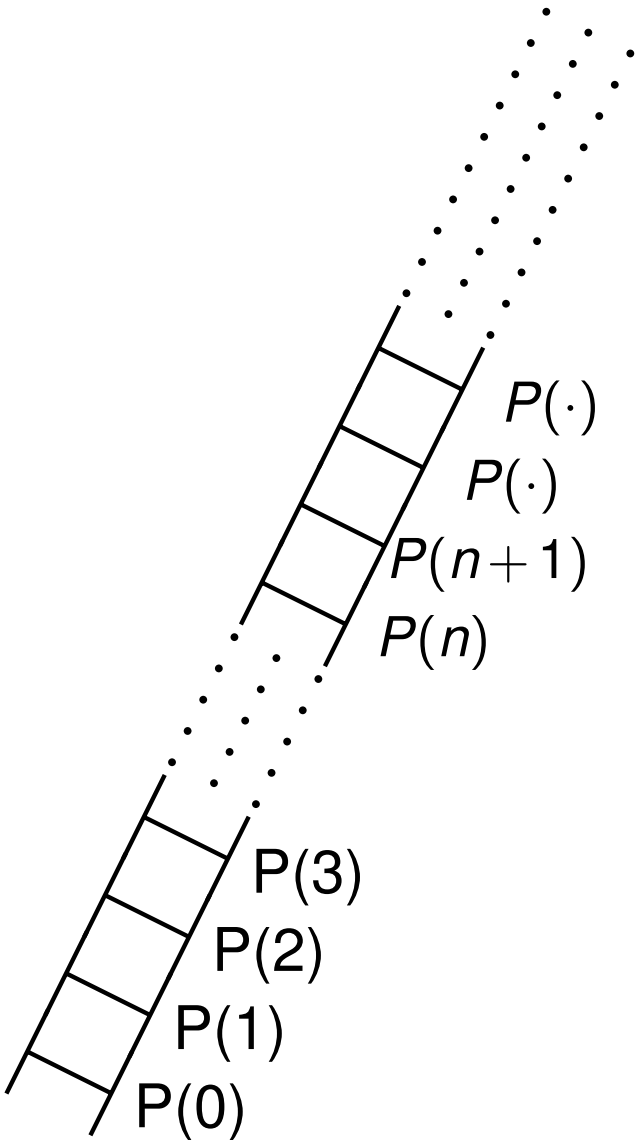

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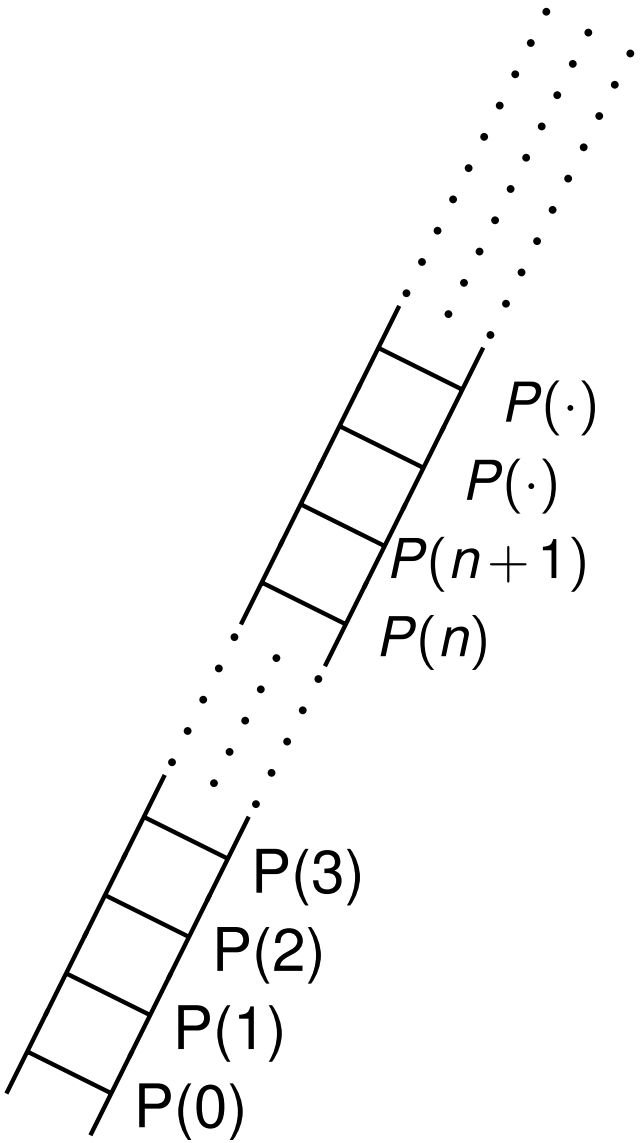

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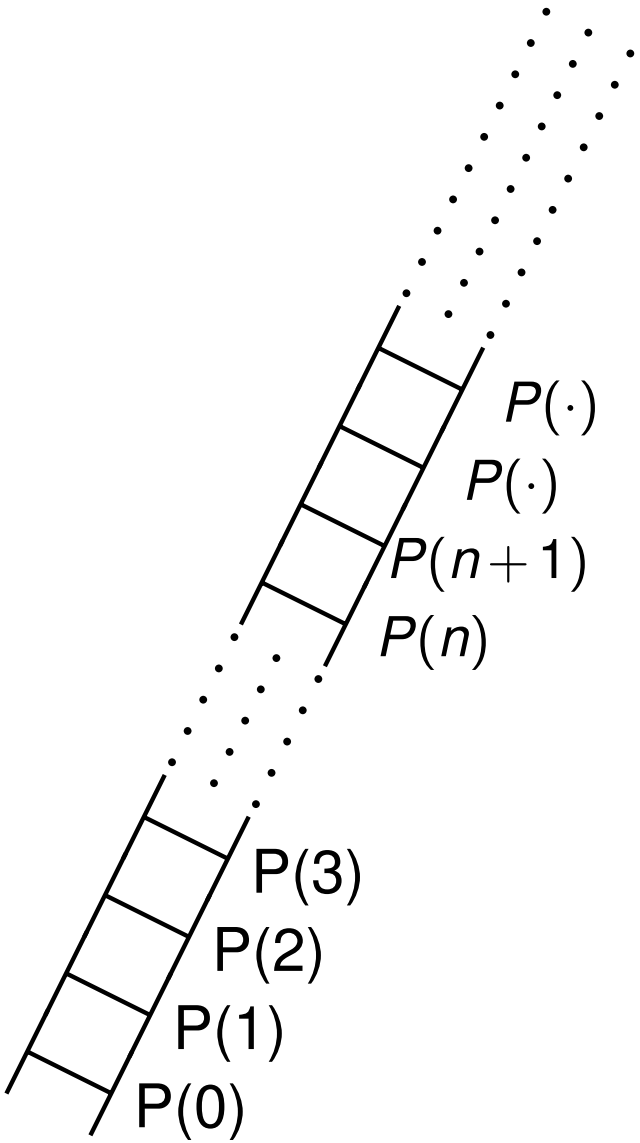

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Theorem: For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$

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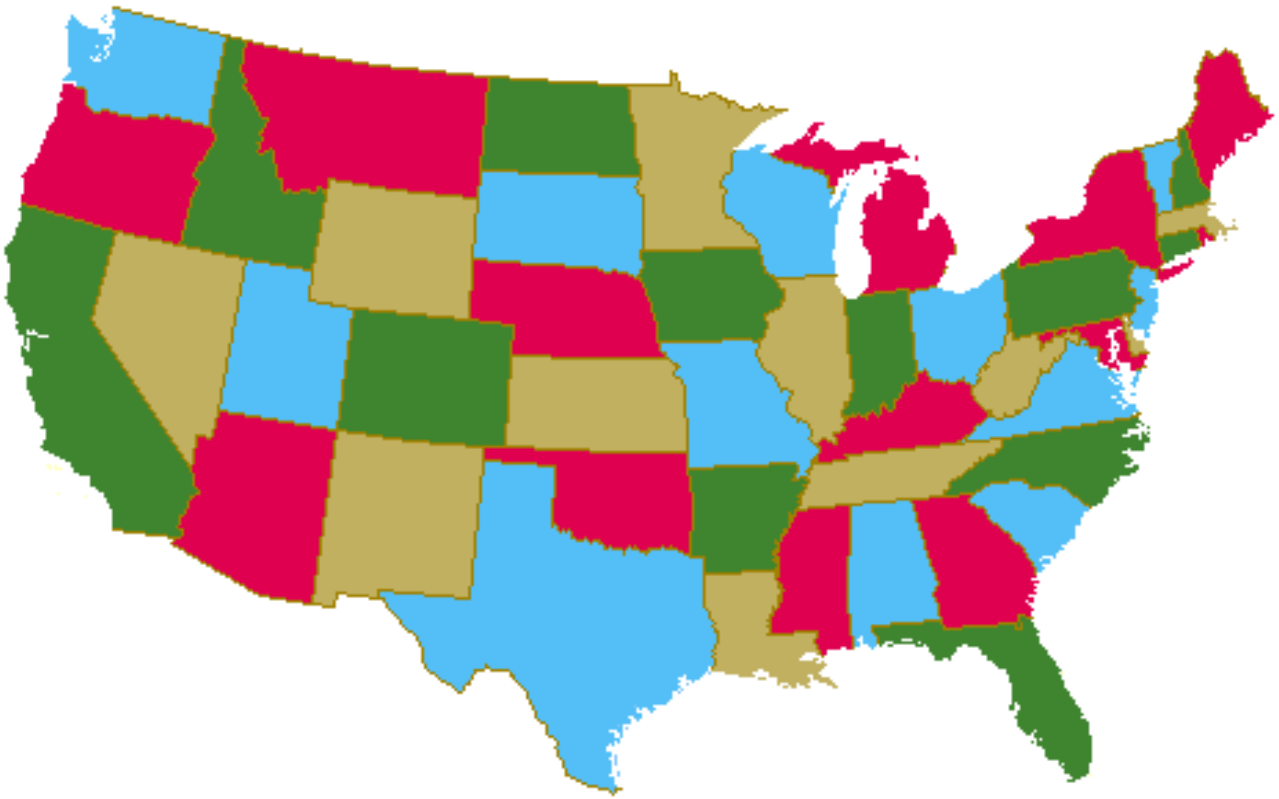
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



Two color theorem: example.

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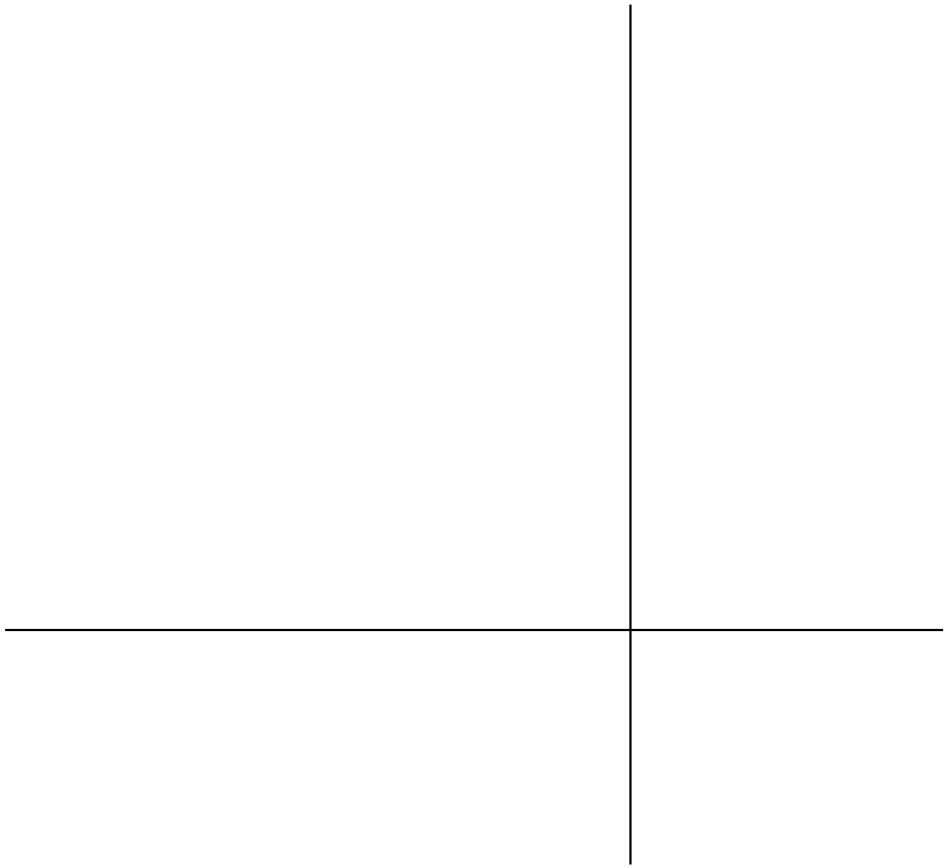

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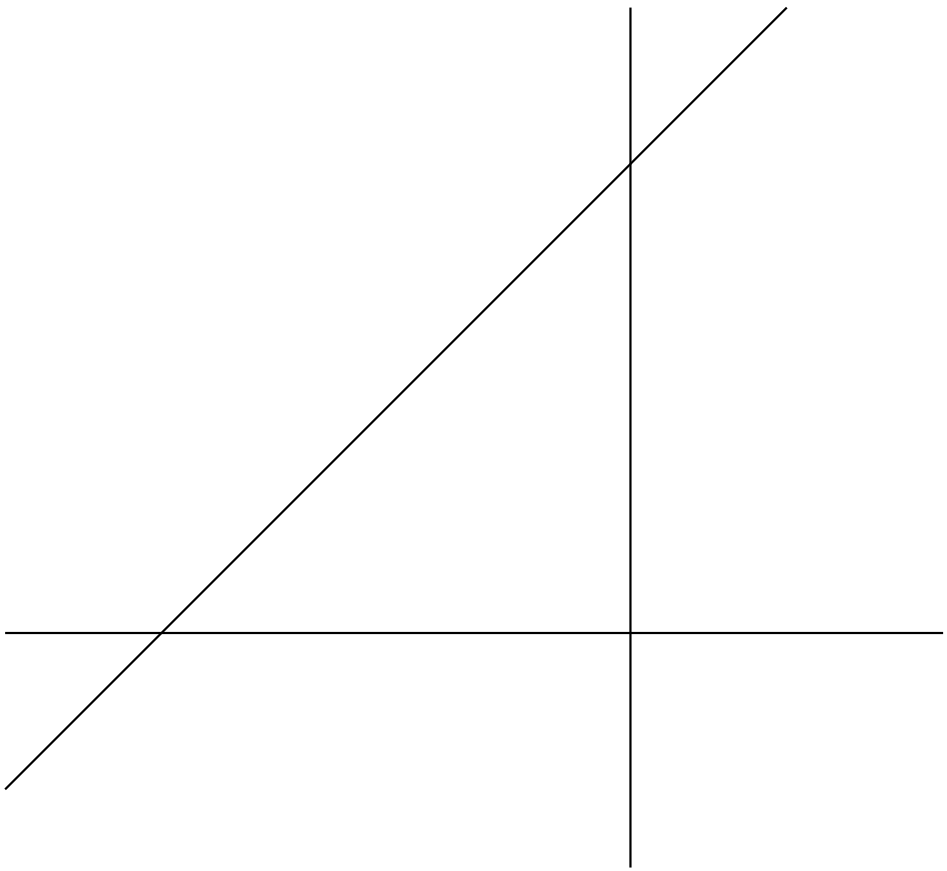

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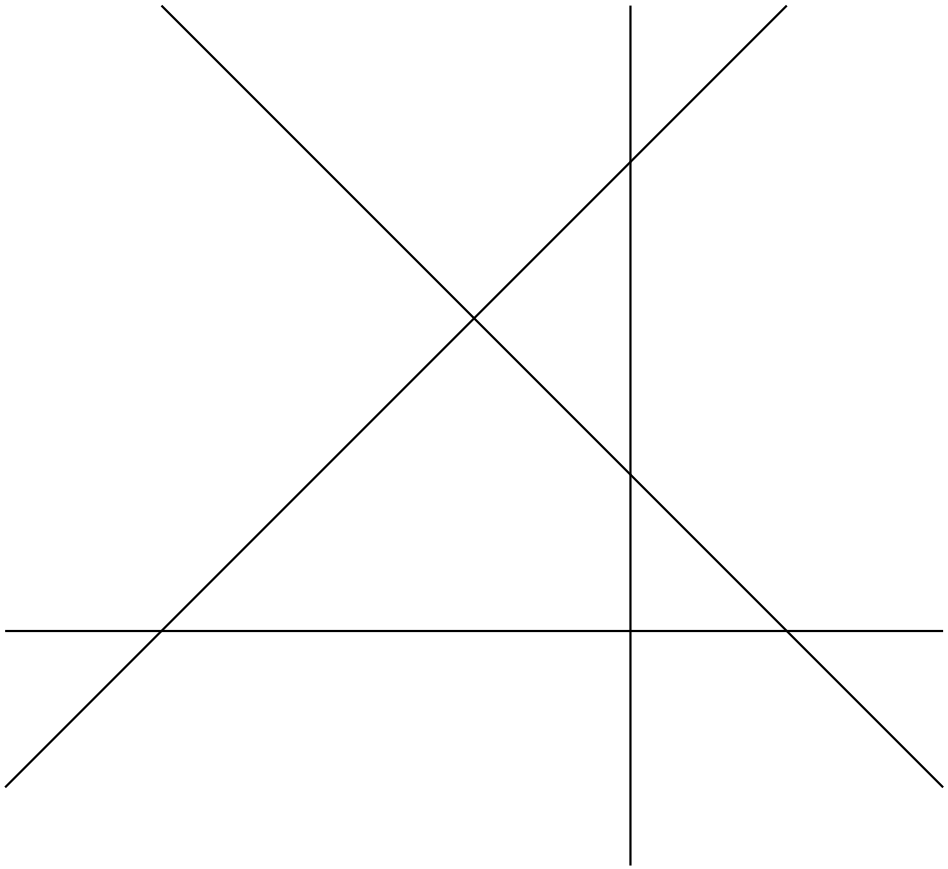

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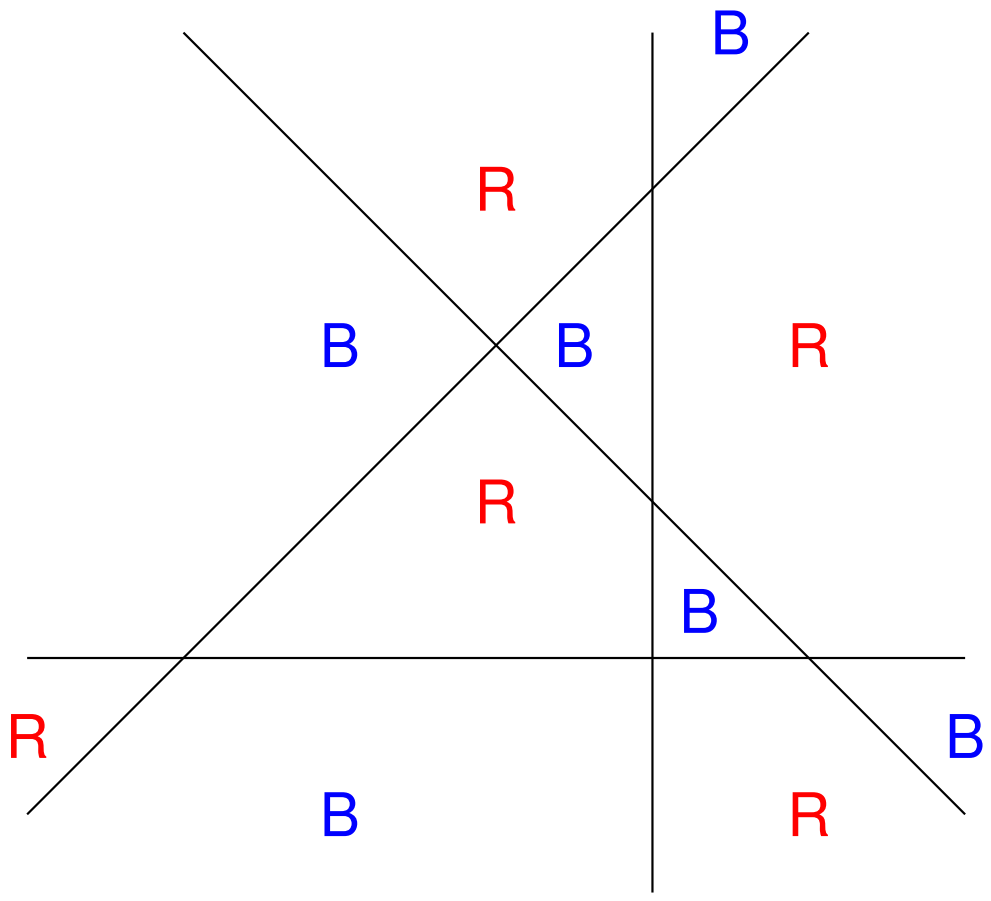

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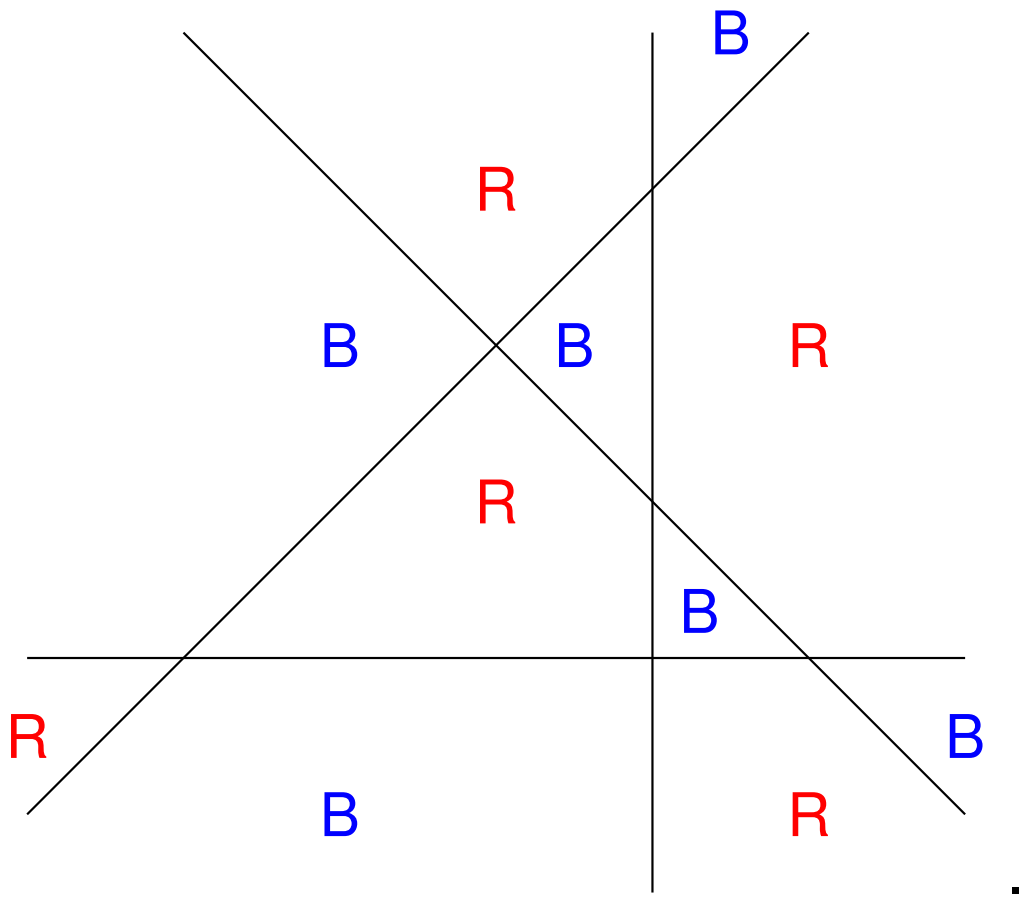

Fact: Swapping red and blue gives another valid colors.

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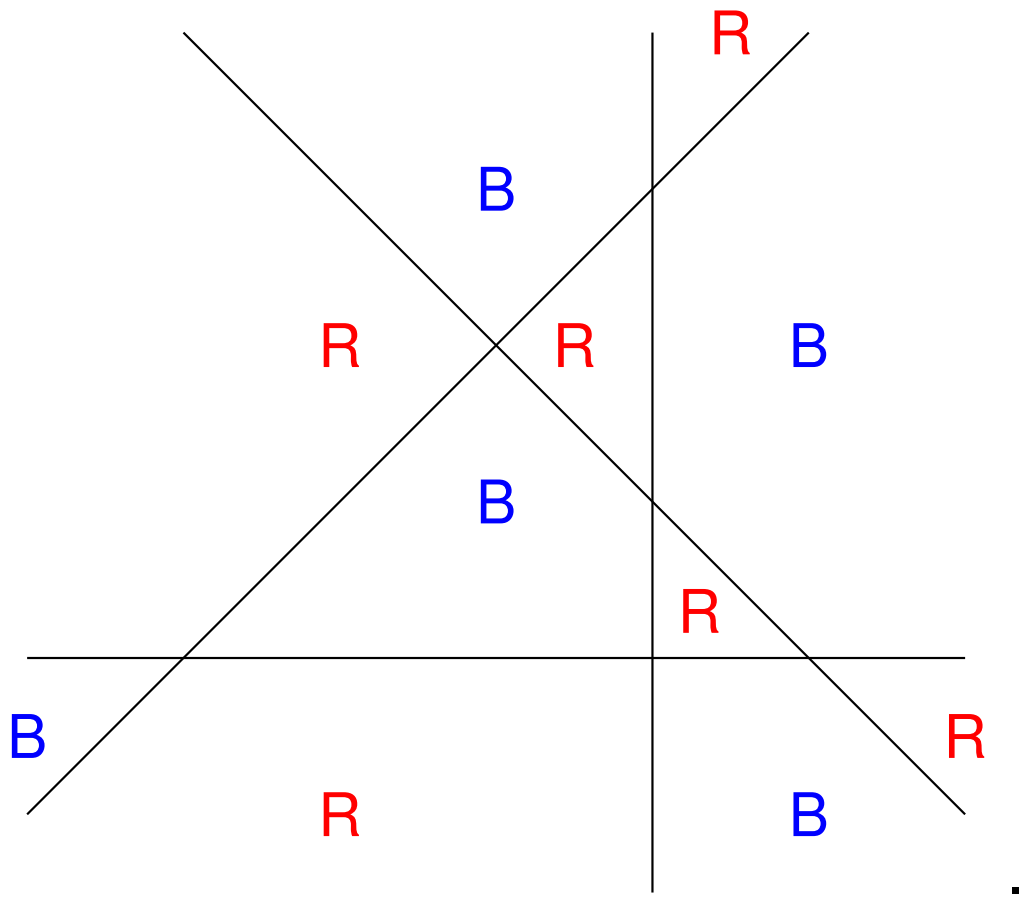

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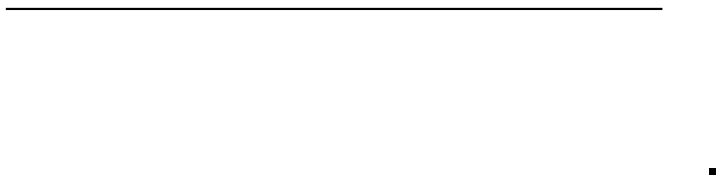

Base Case.

Two color theorem: proof illustration.

R

B

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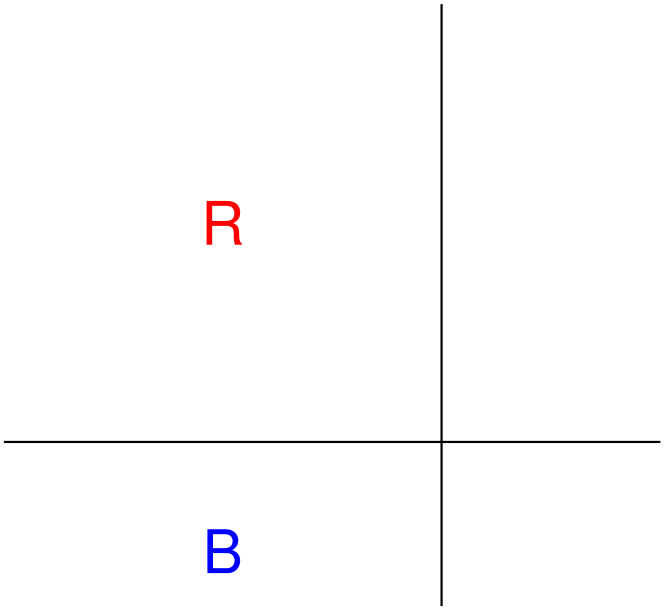
Two color theorem: proof illustration.



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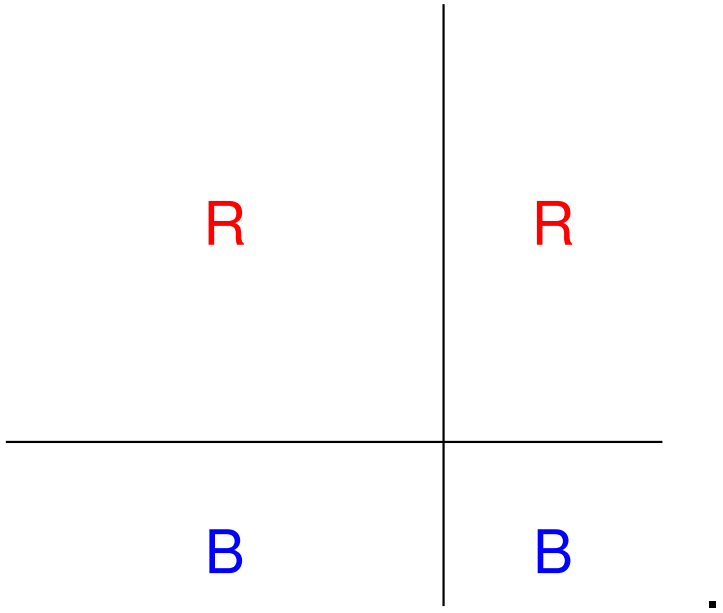
1. Add line.

Two color theorem: proof illustration.



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2. Get inherited color for split regions

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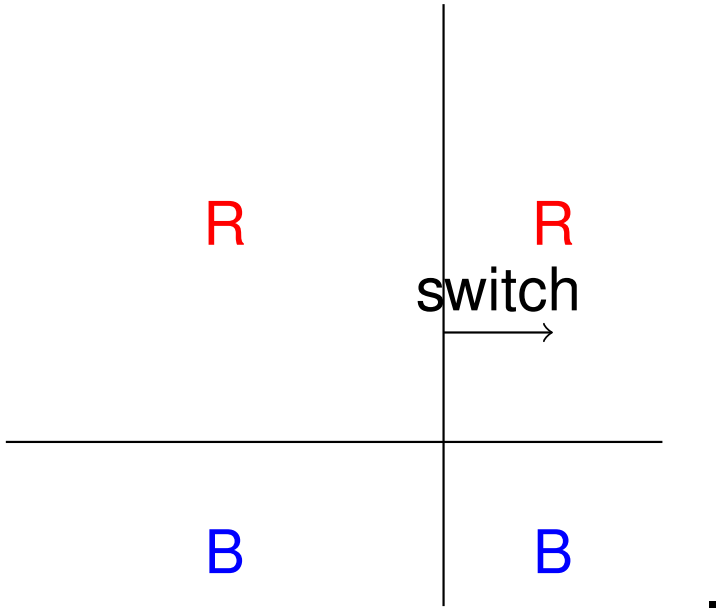

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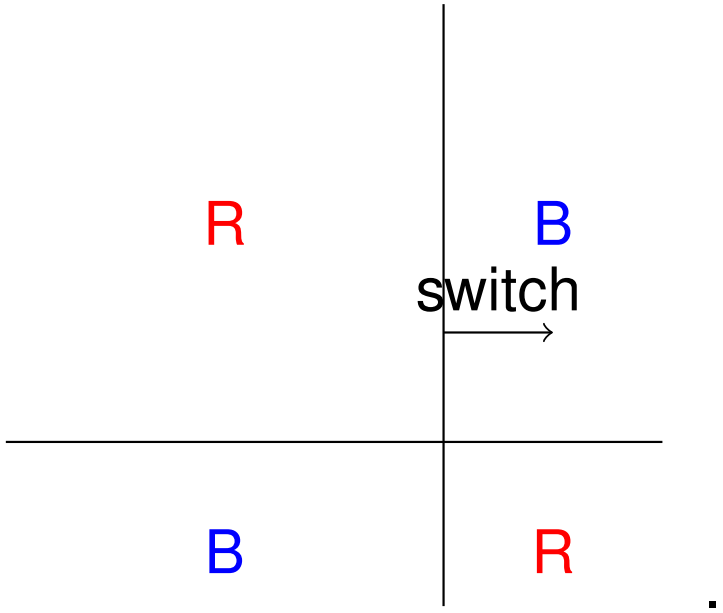

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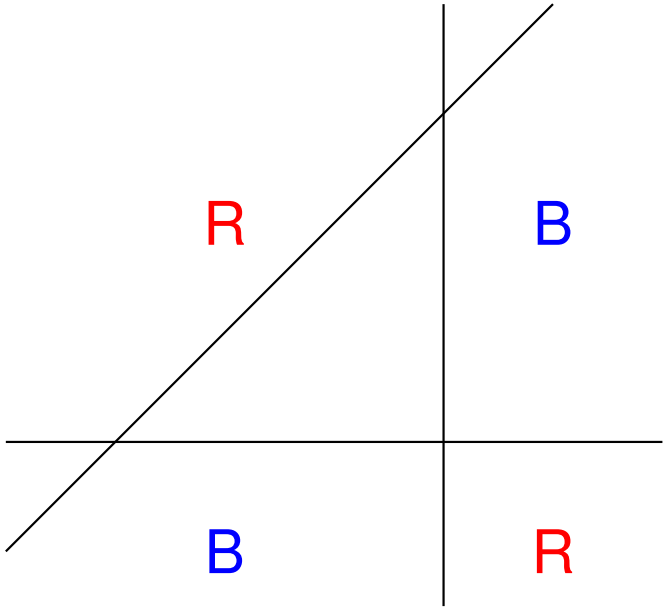

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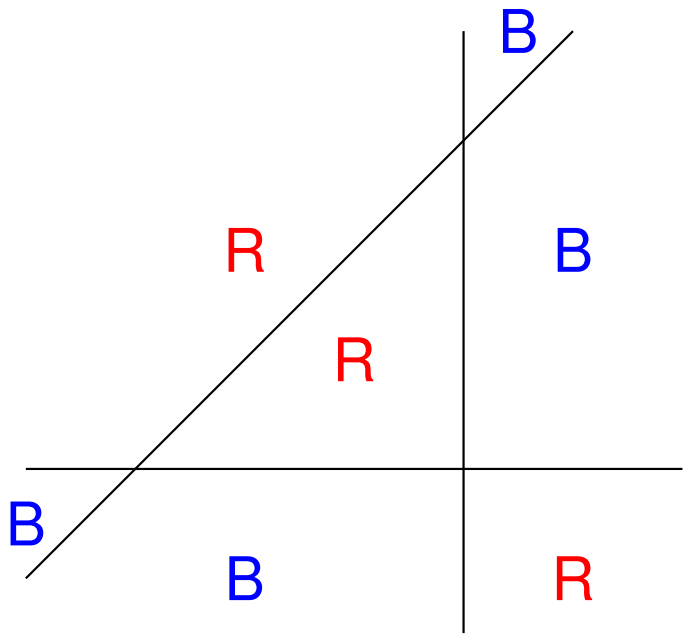

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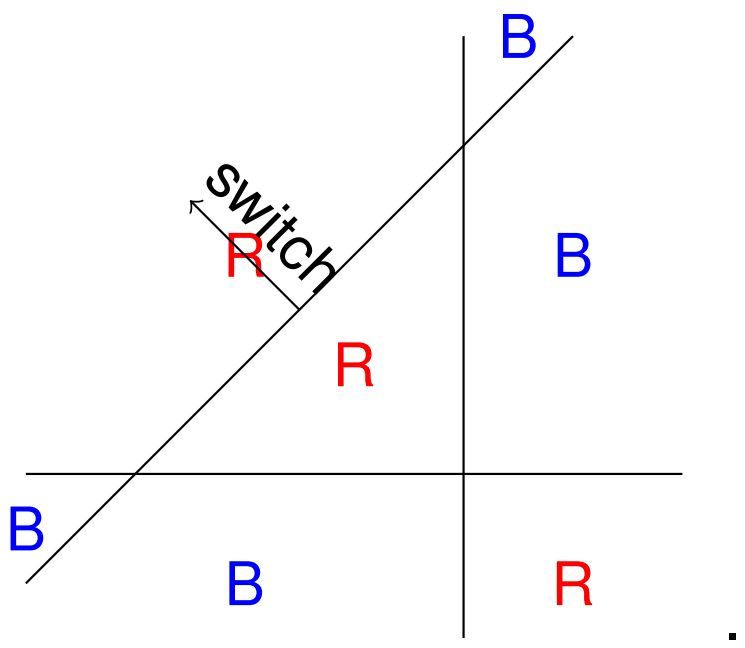

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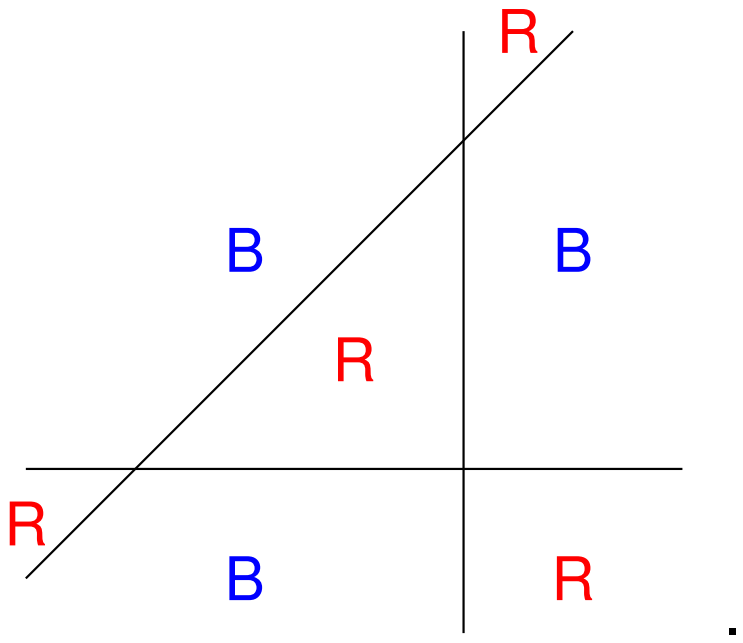

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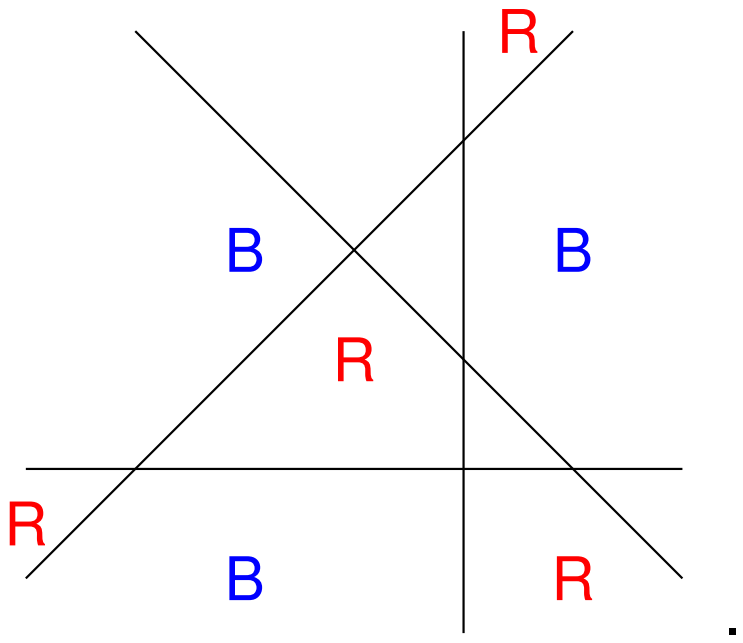

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2. Get inherited color for split regions
3. Switch on one side of new line.
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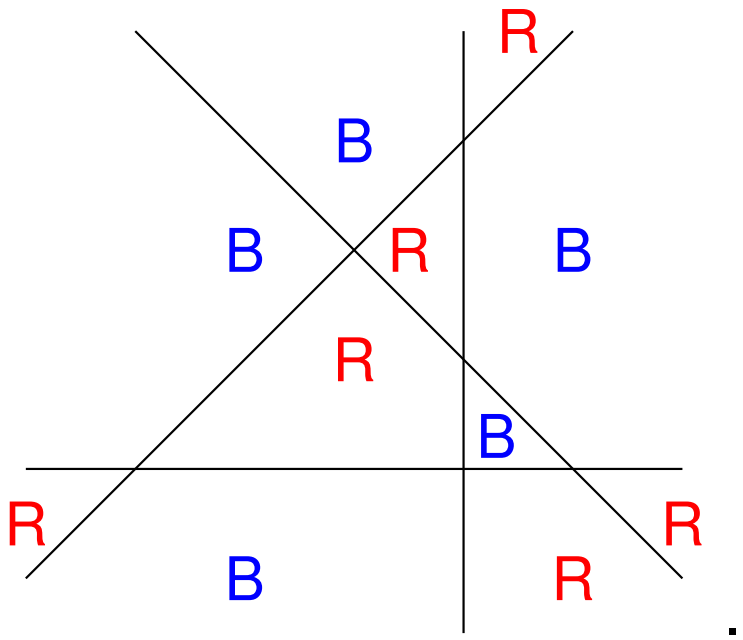

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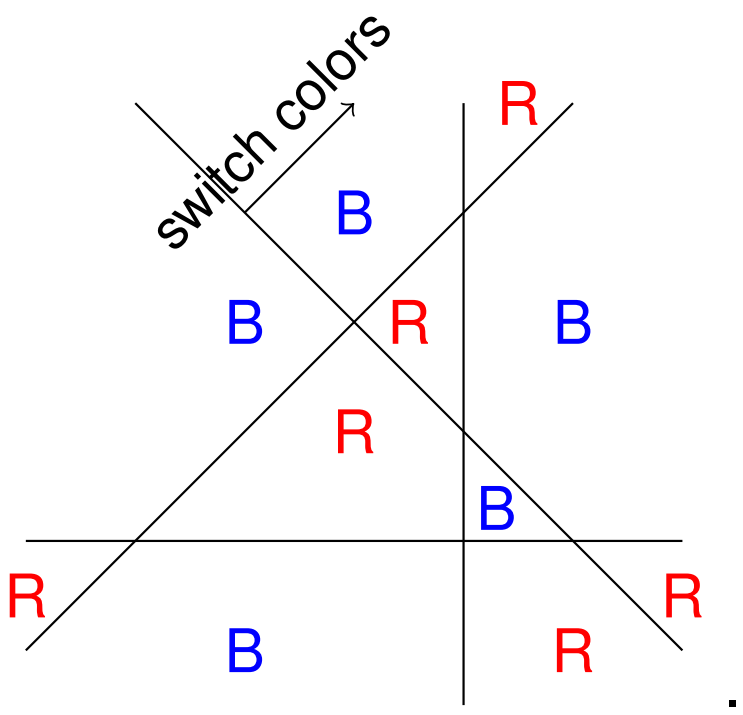

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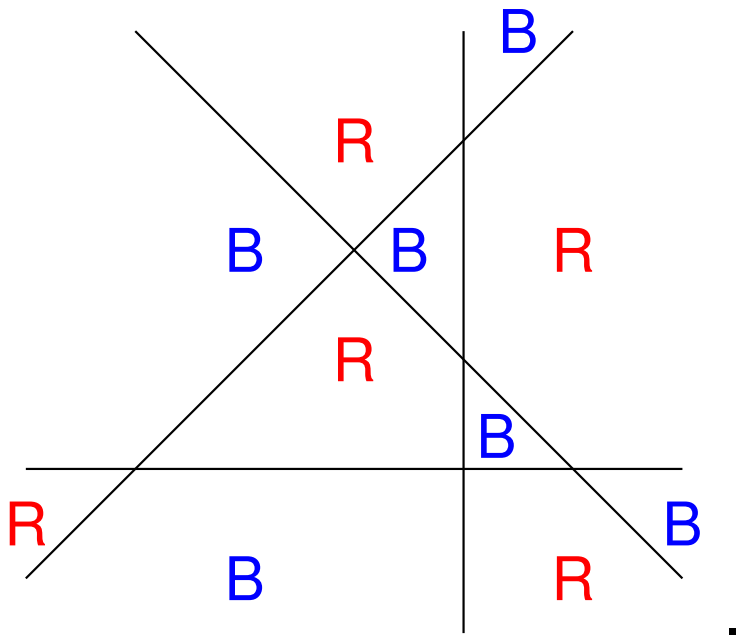

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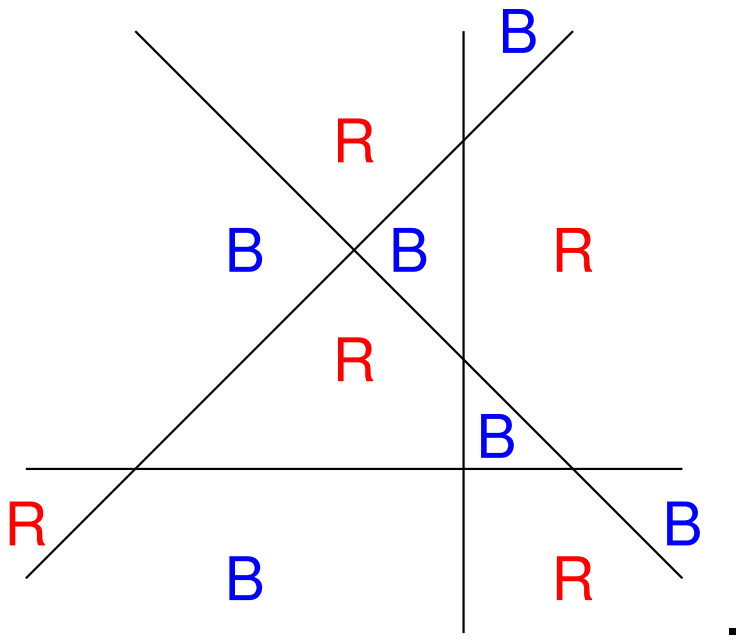

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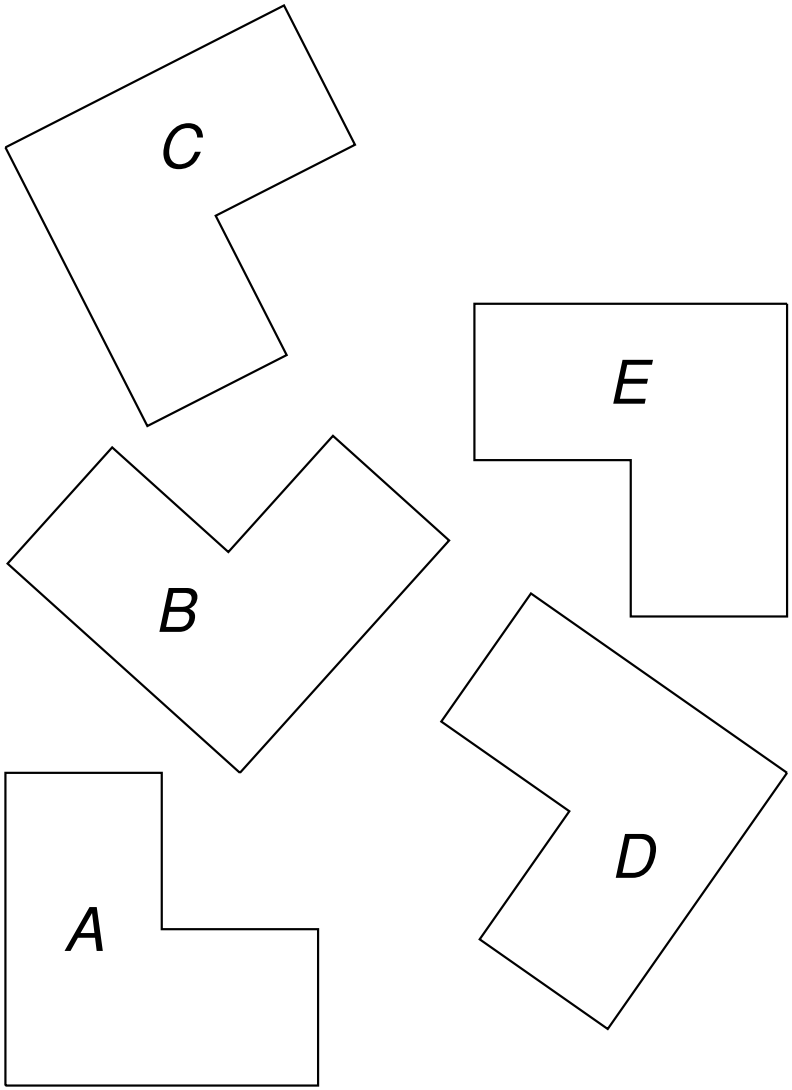
Algorithm gives $P(k) \implies P(k+1)$.

Tiling Cory Hall Courtyard.



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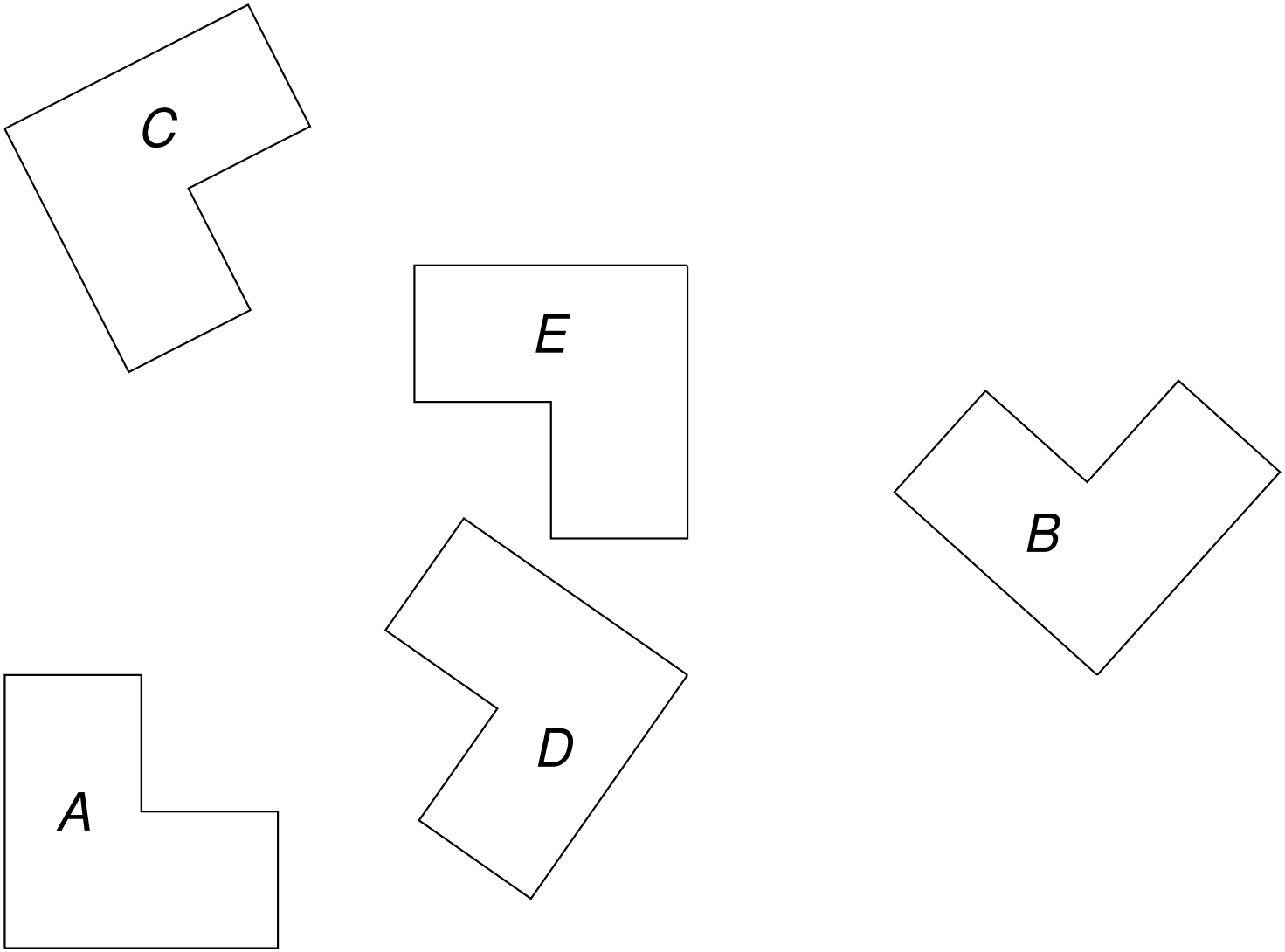

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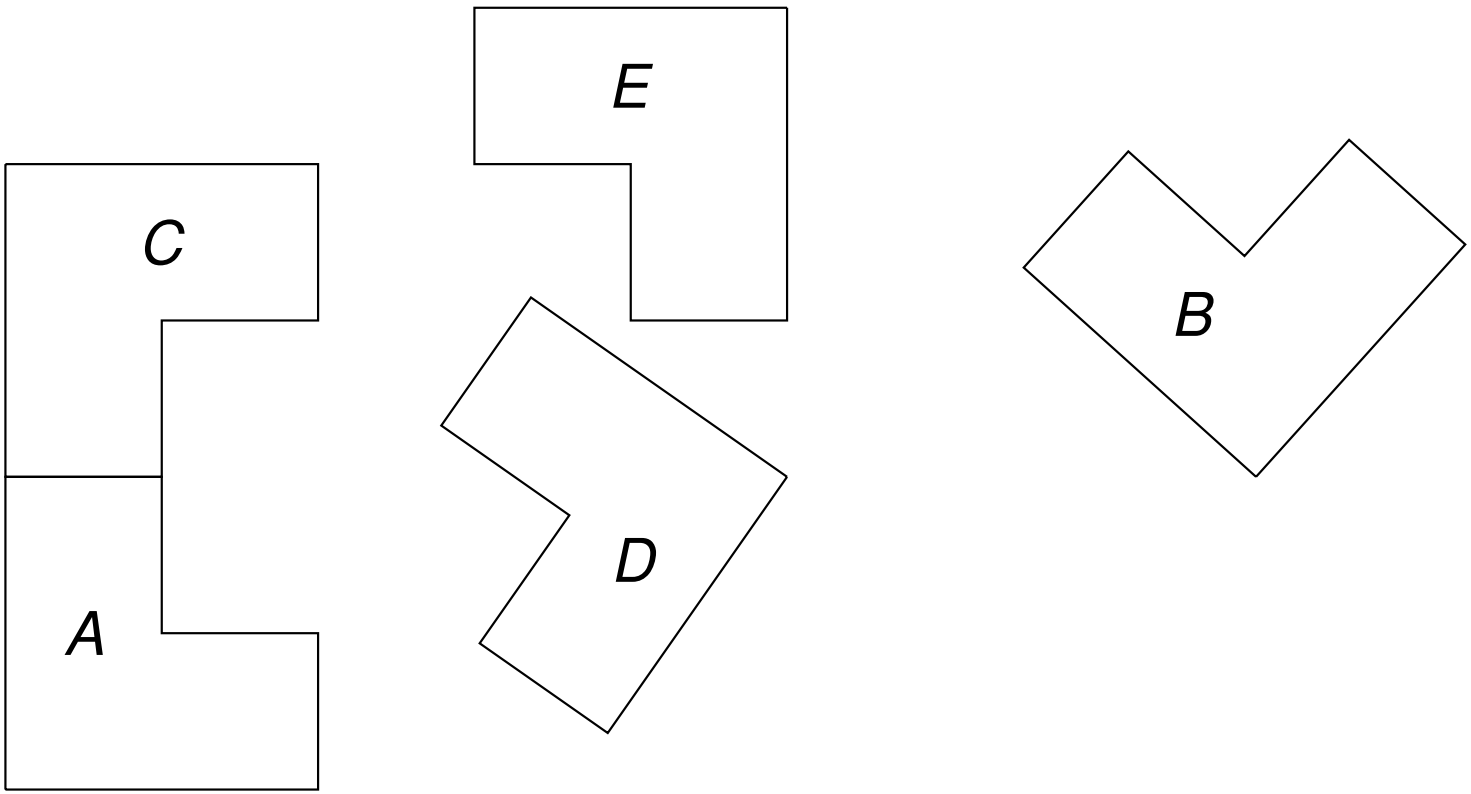

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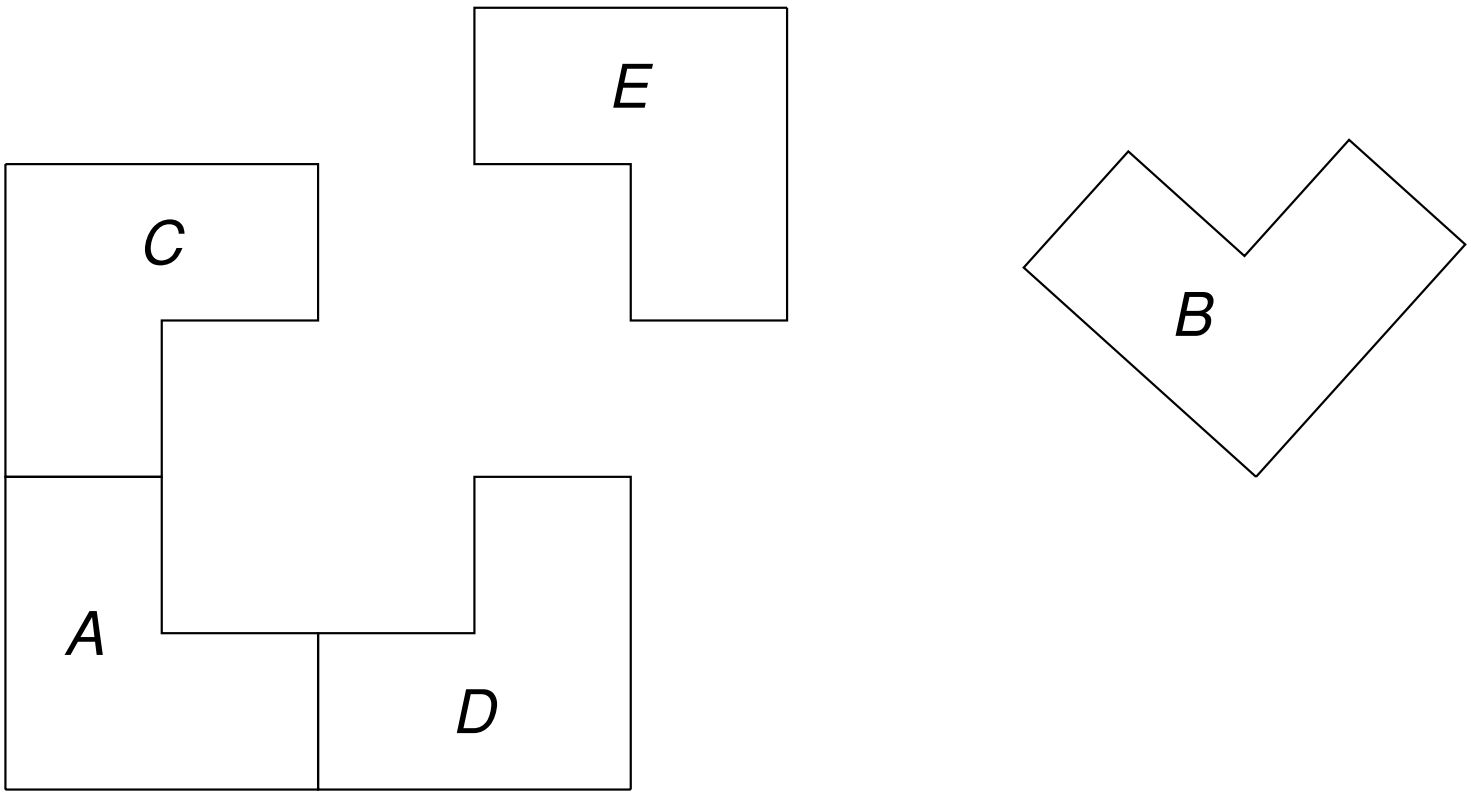

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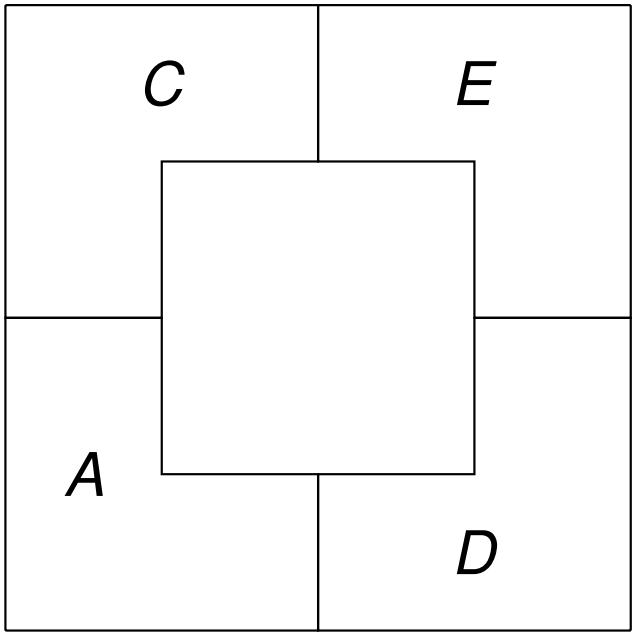

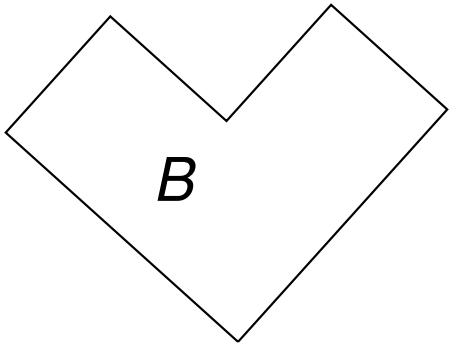

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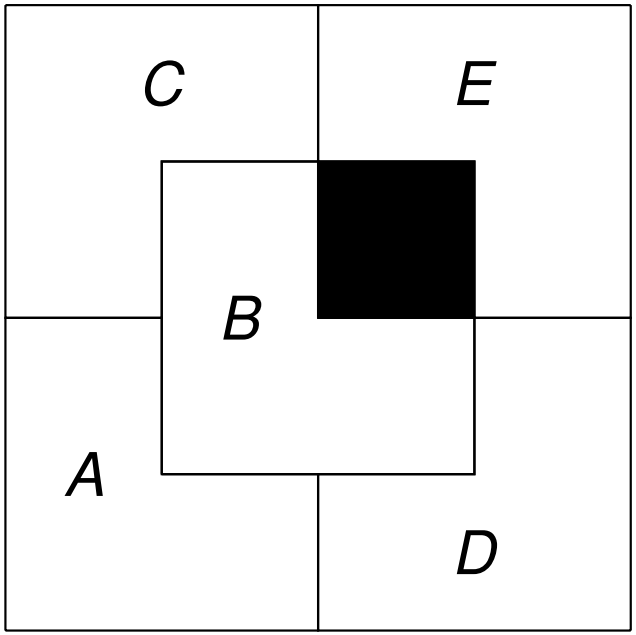

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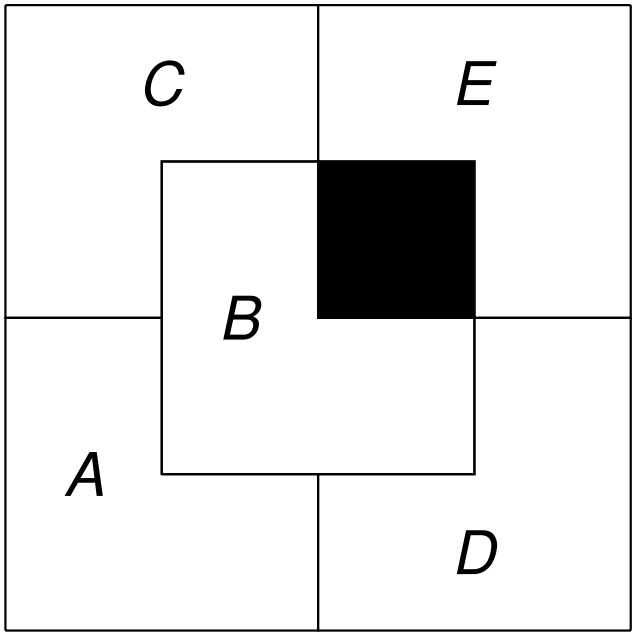

Summary: principle of induction.

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Statement to prove: $P(n)$ for n starting from n_0

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Statement is proven!