CS70: Jean Walrand: Lecture 34.

Conditional Expectation

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Conditional Expectation

1. Review: joint distribution, LLSE
2. Definition of Conditional expectation
3. Properties of CE
4. Applications: Diluting, Mixing, Rumors
5. CE = MMSE

Review

Definitions Let X and Y be RVs on Ω.

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Recall the non-Bayesian and Bayesian viewpoints.

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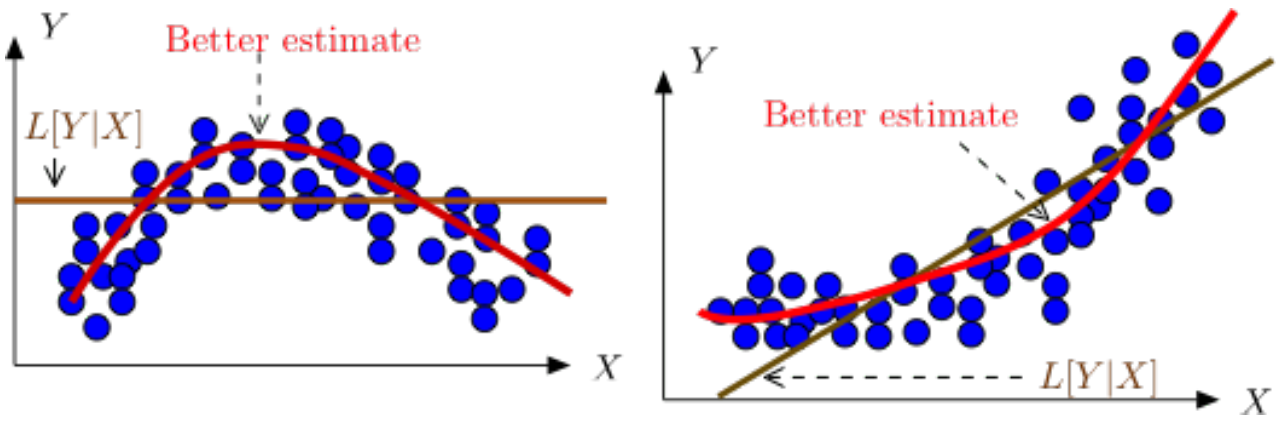

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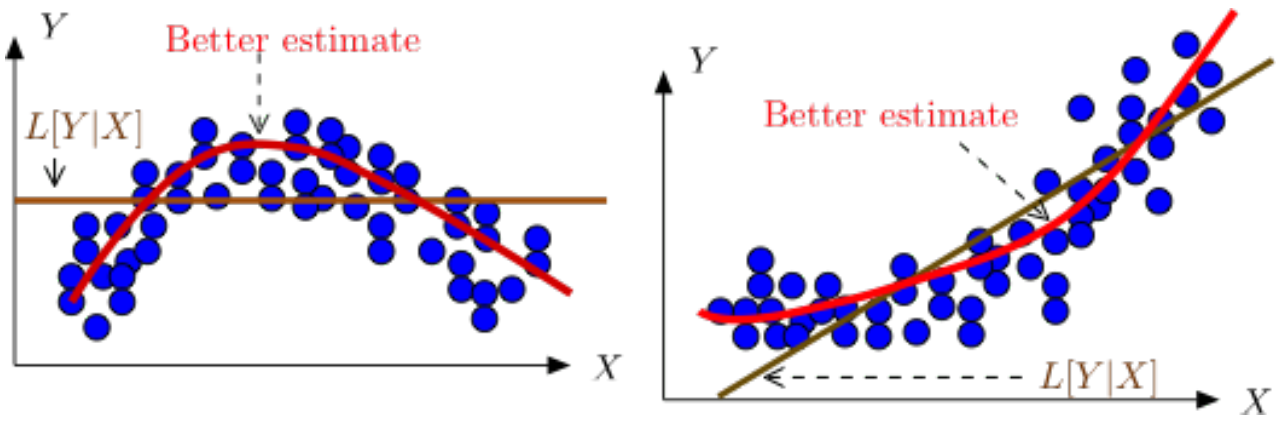

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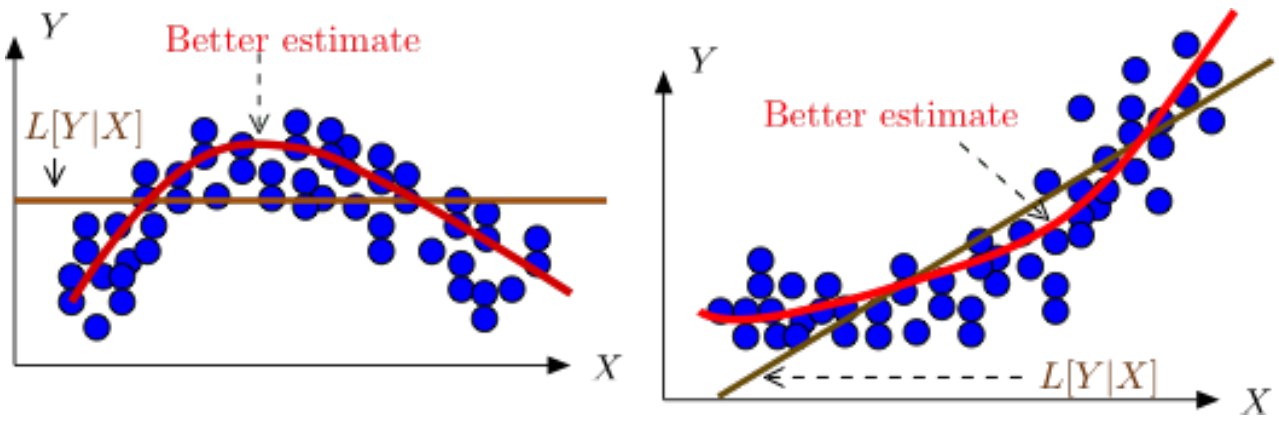

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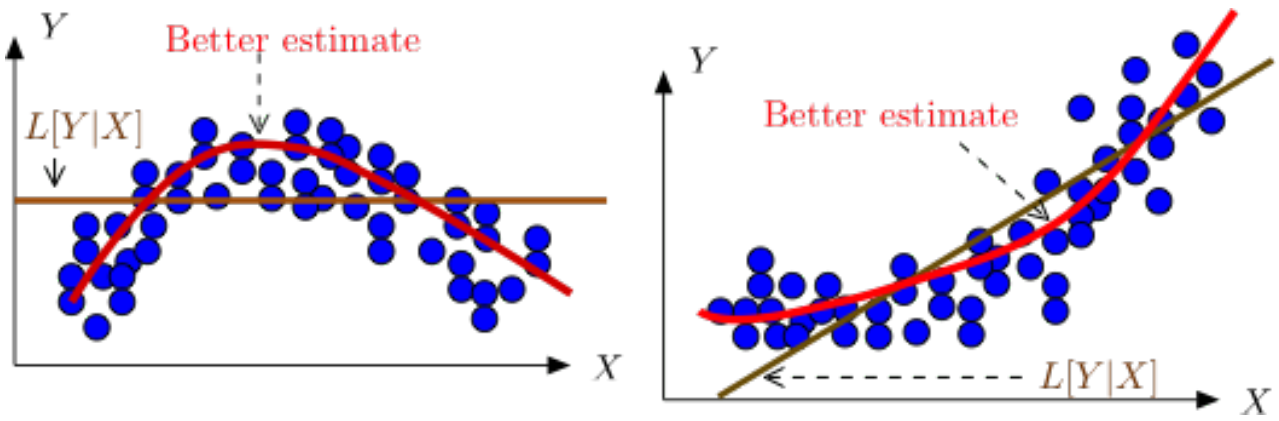

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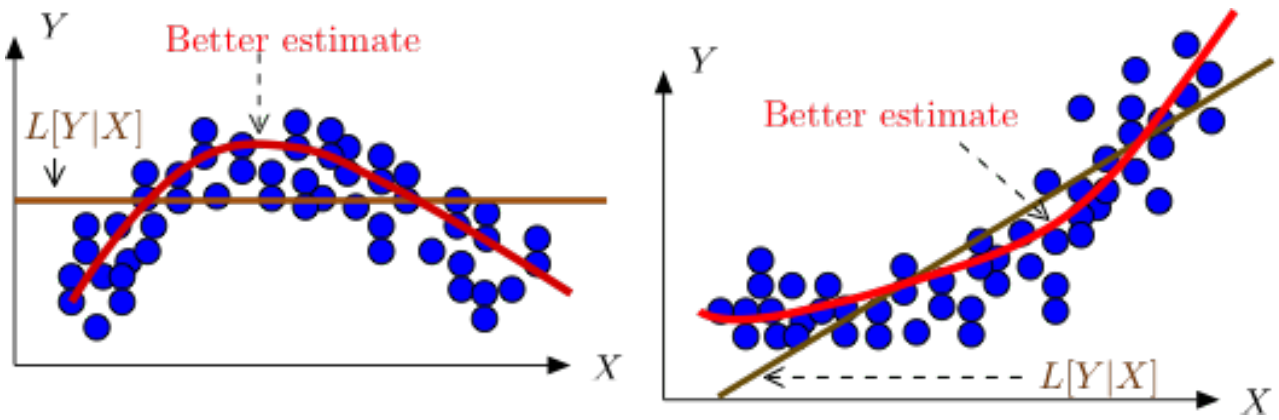

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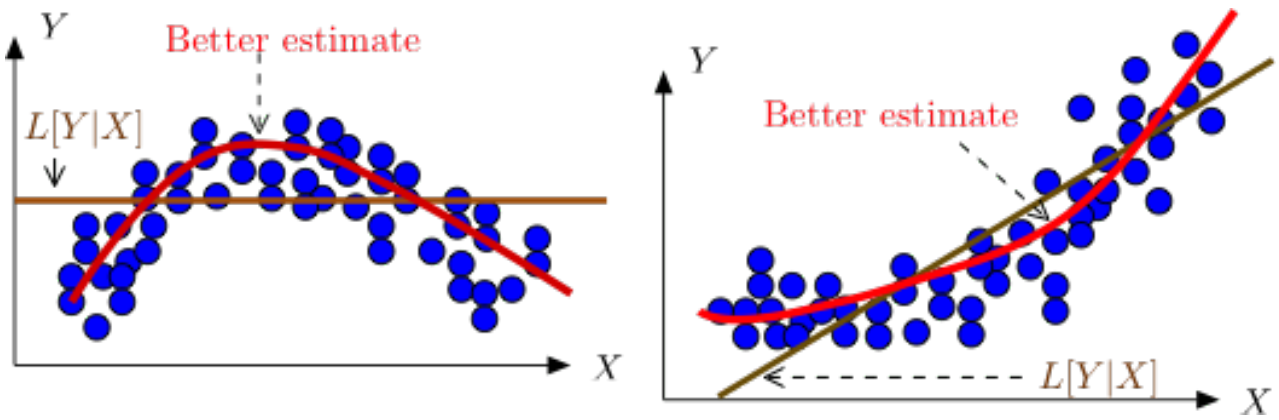

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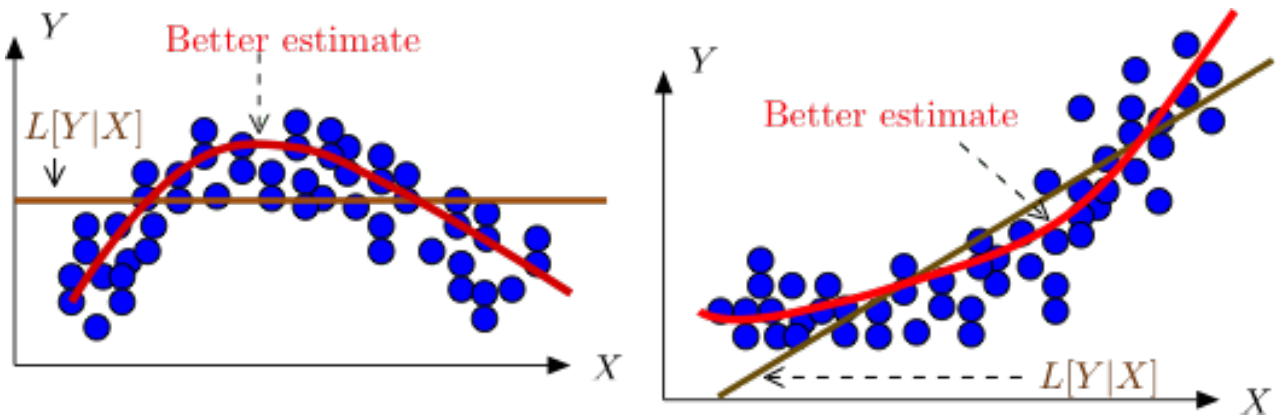

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Proof: $E[Y|X = x] = E[Y|A]$ with $A = \{\omega : X(\omega) = x\}$. $\square$

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(e) Let $h(X) = 1$ in (d).

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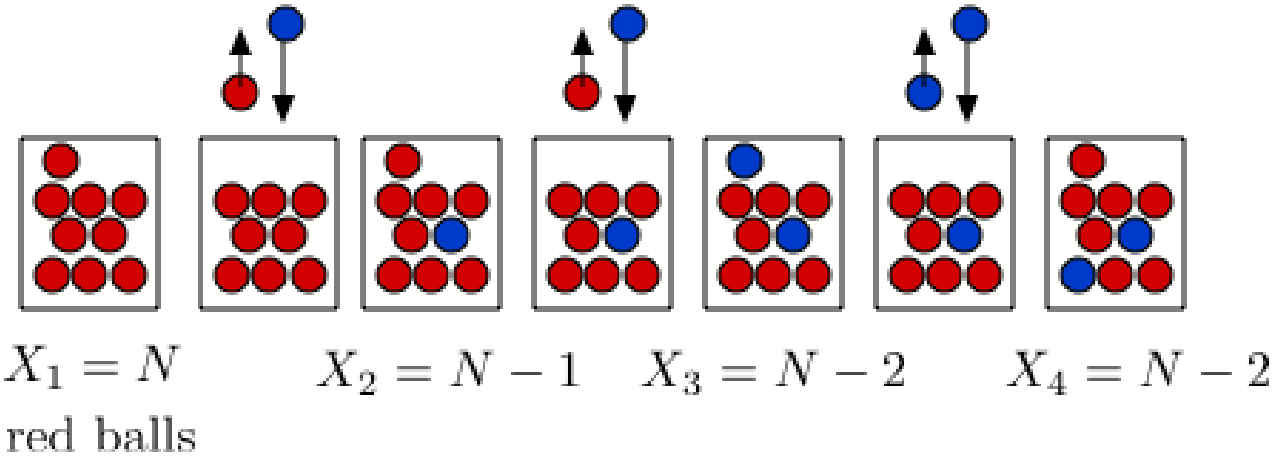

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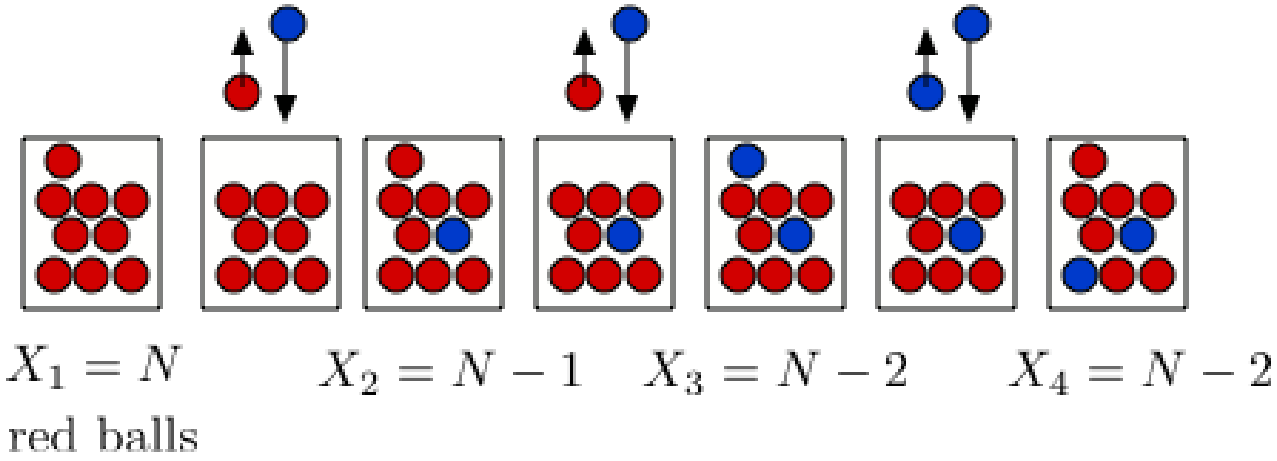

At each step, pick a ball from a well-mixed urn.

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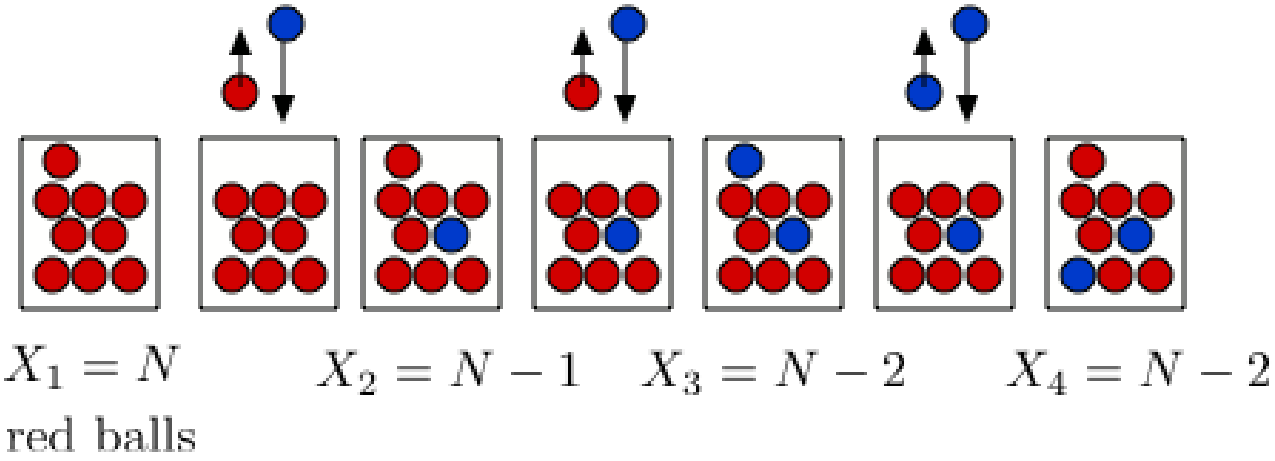

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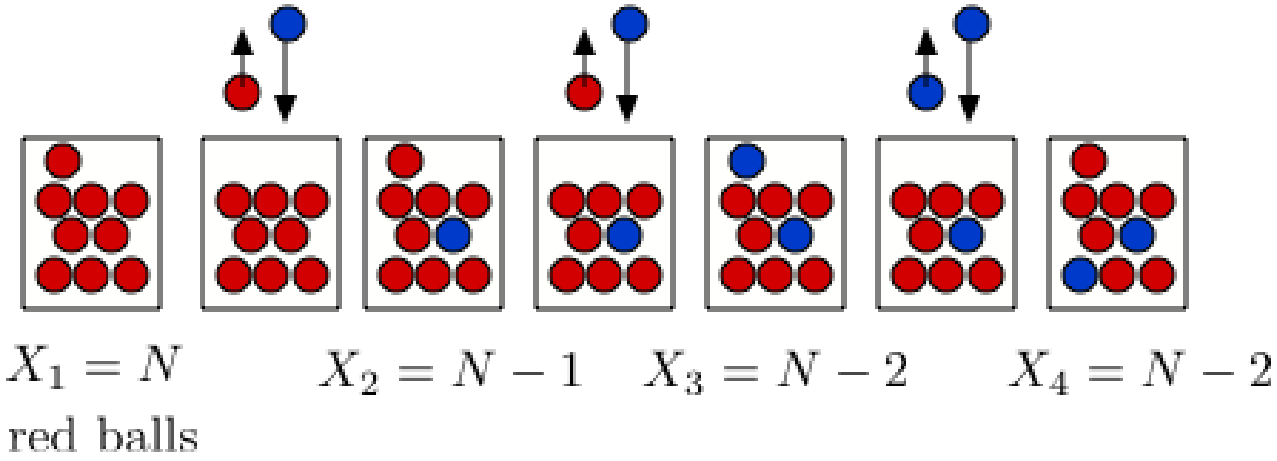

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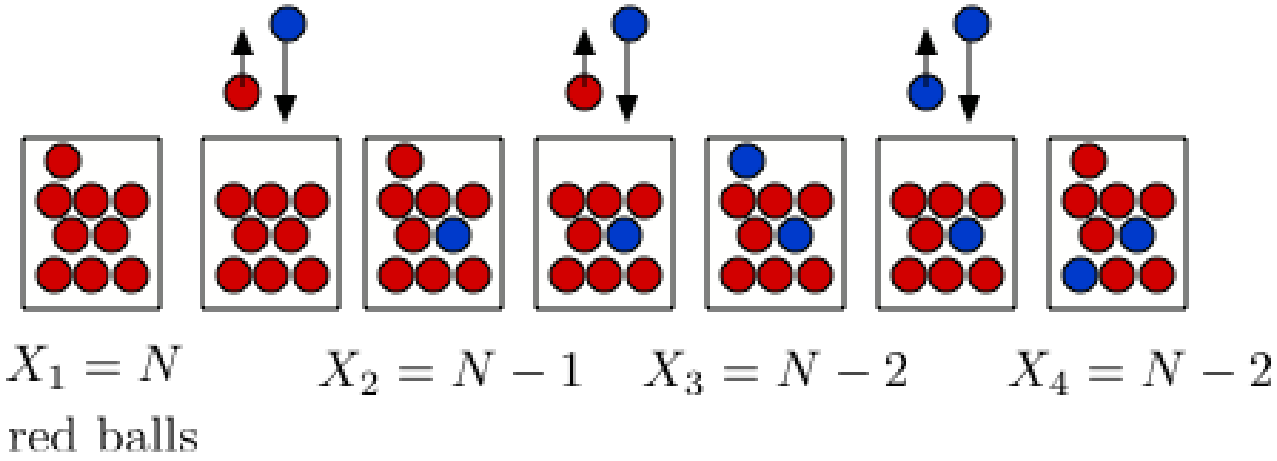

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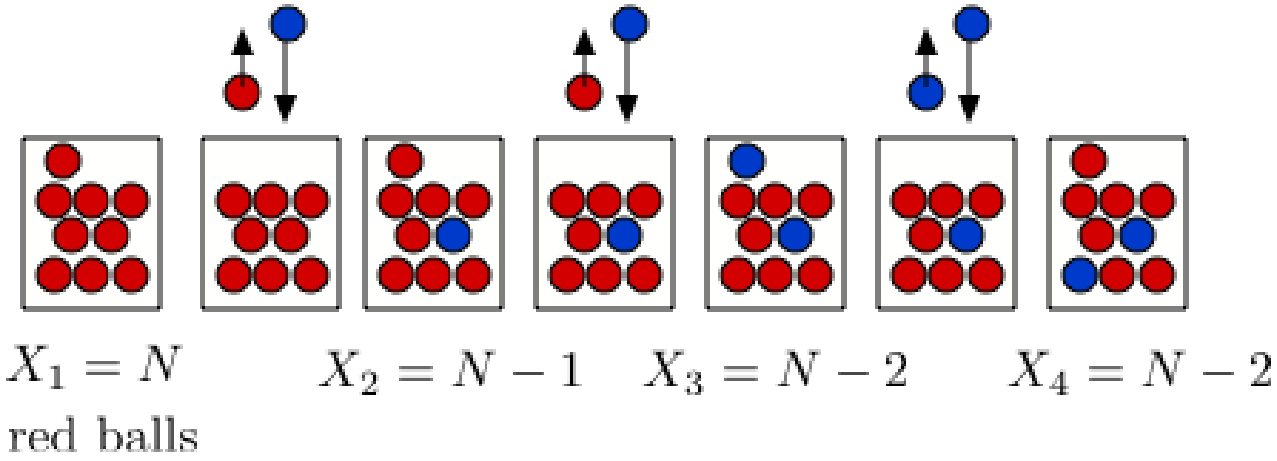

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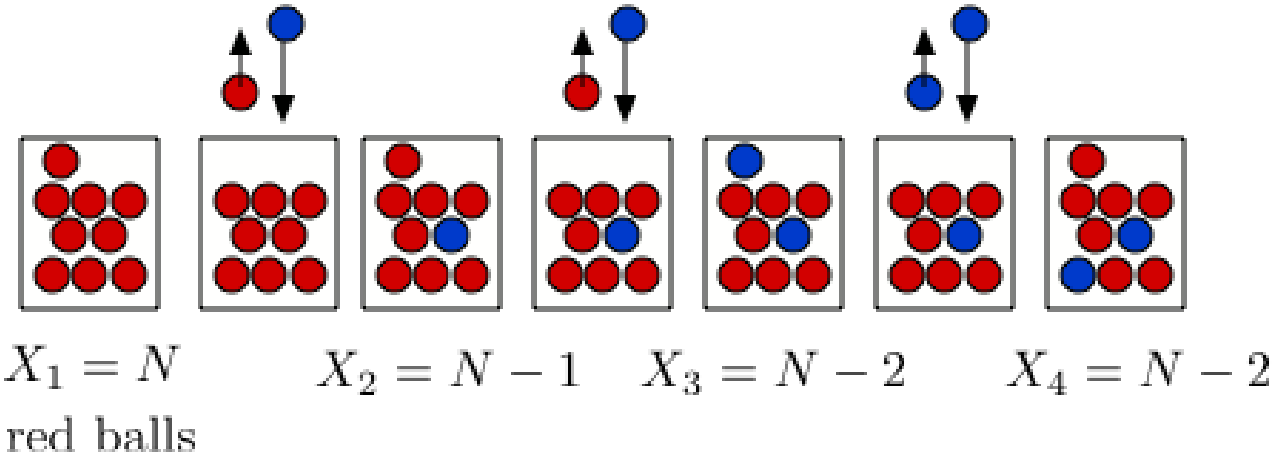

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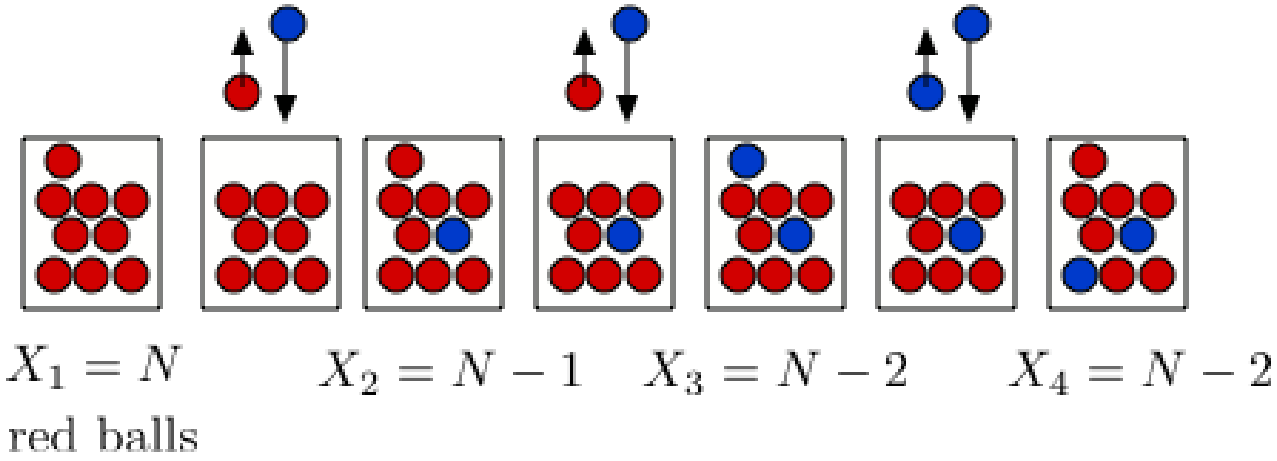

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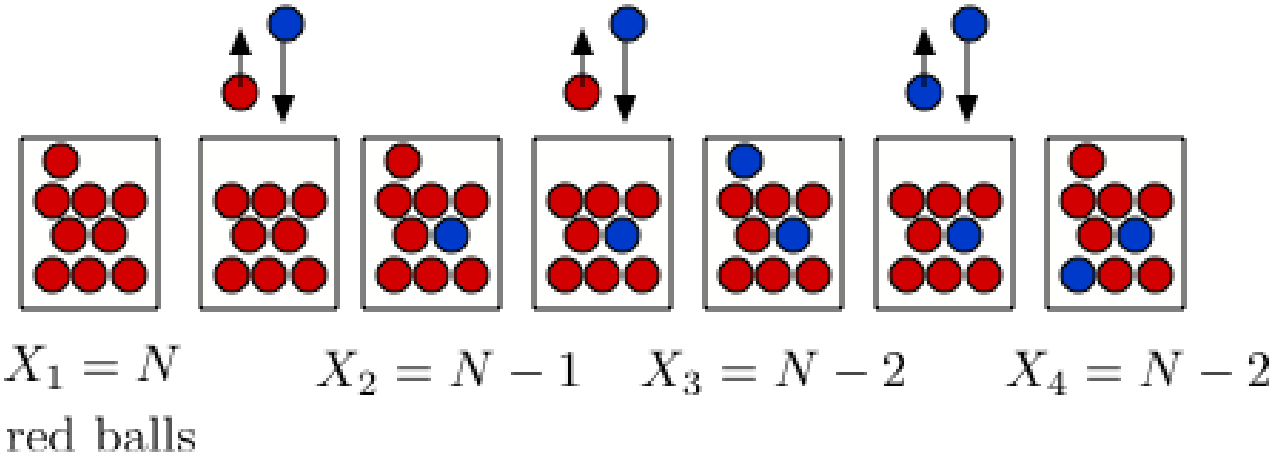

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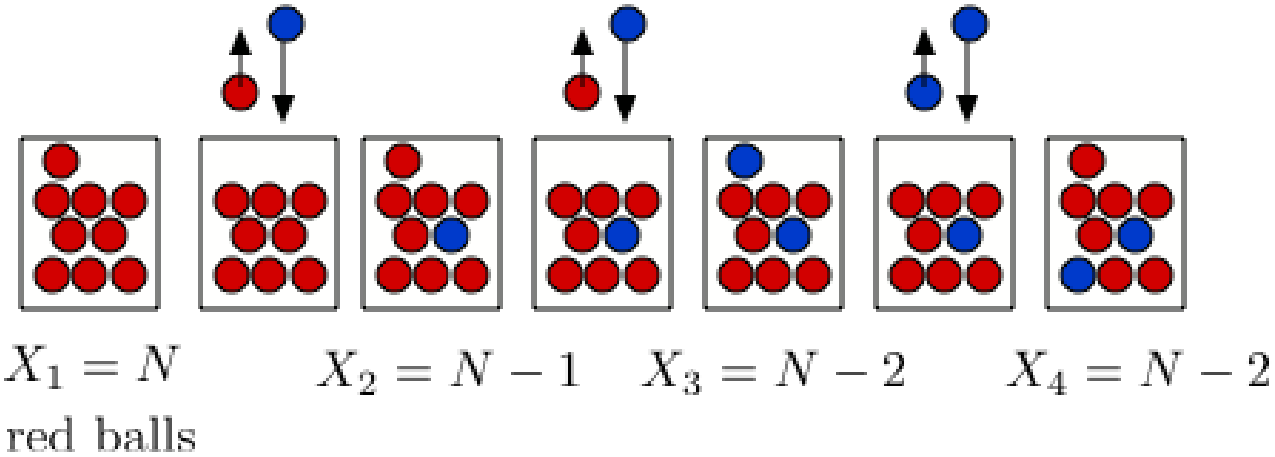

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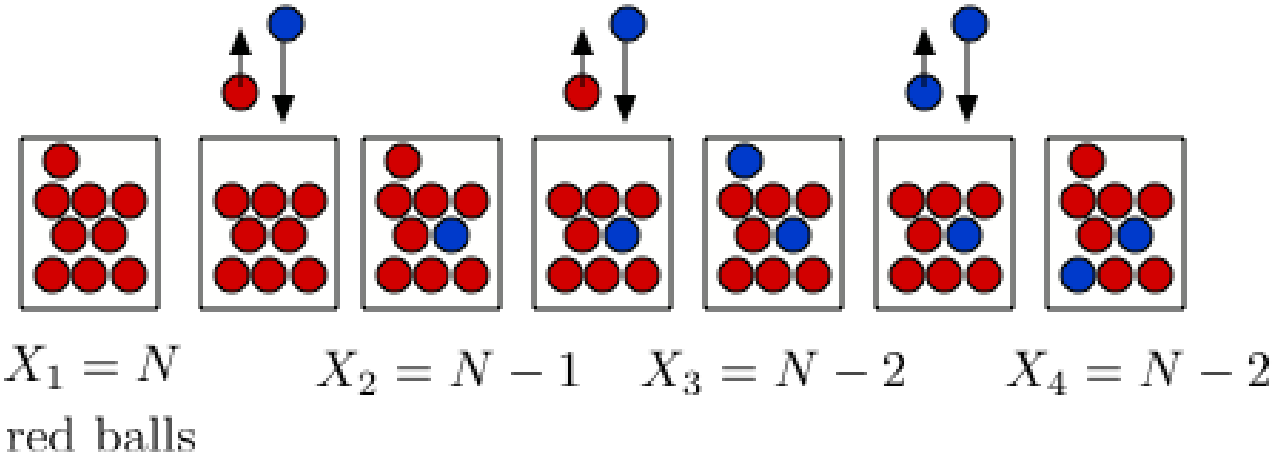

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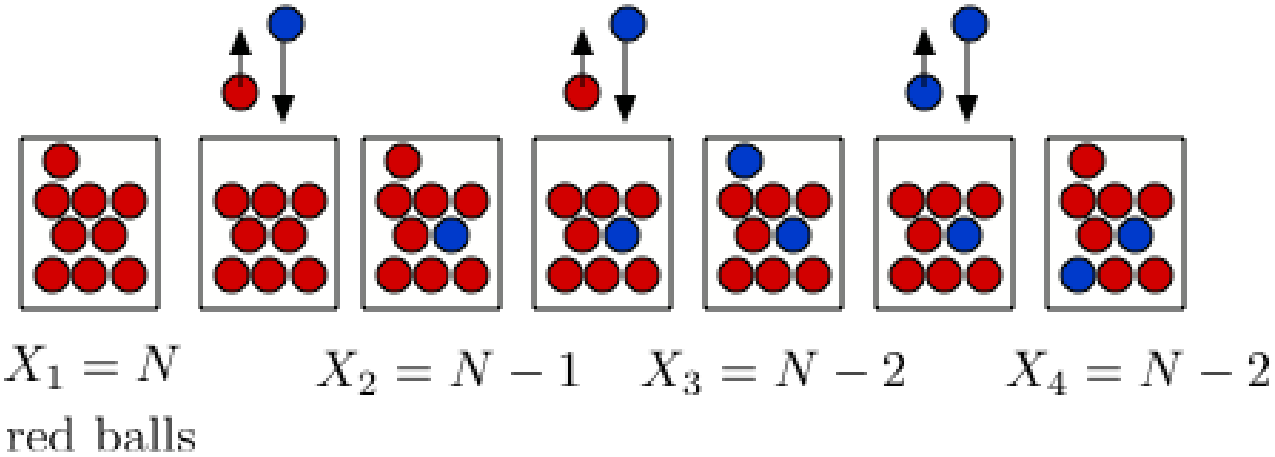

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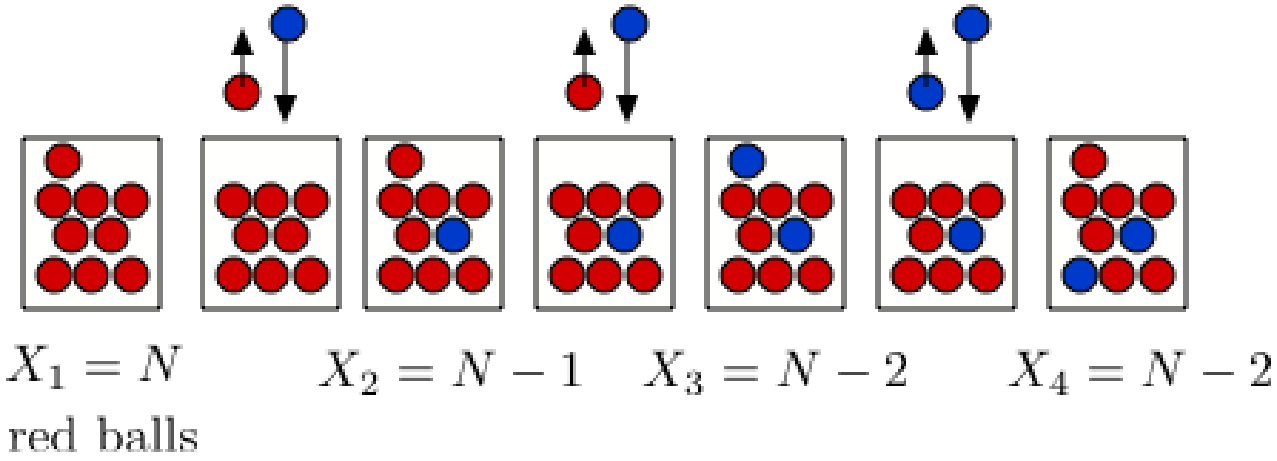

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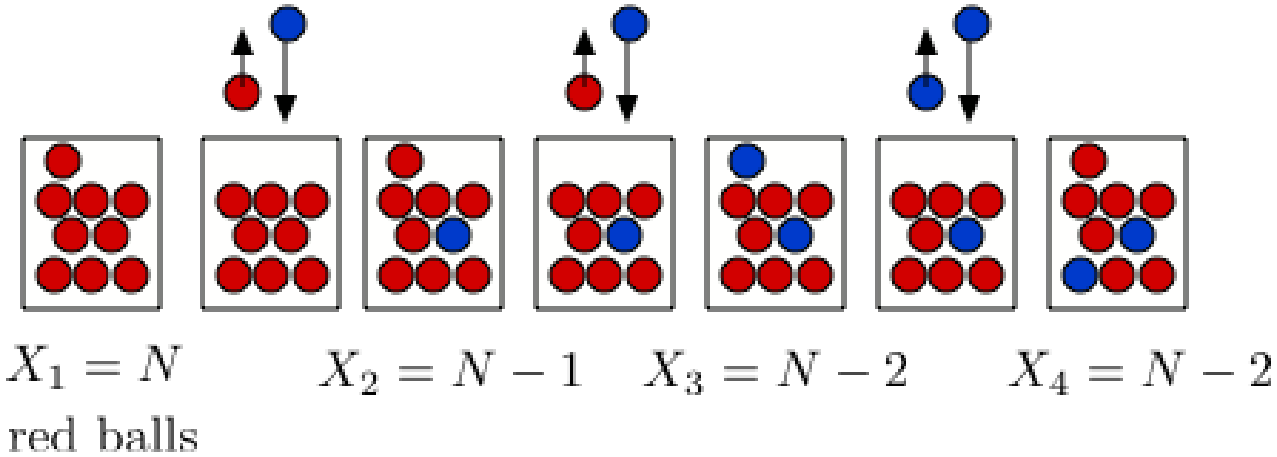

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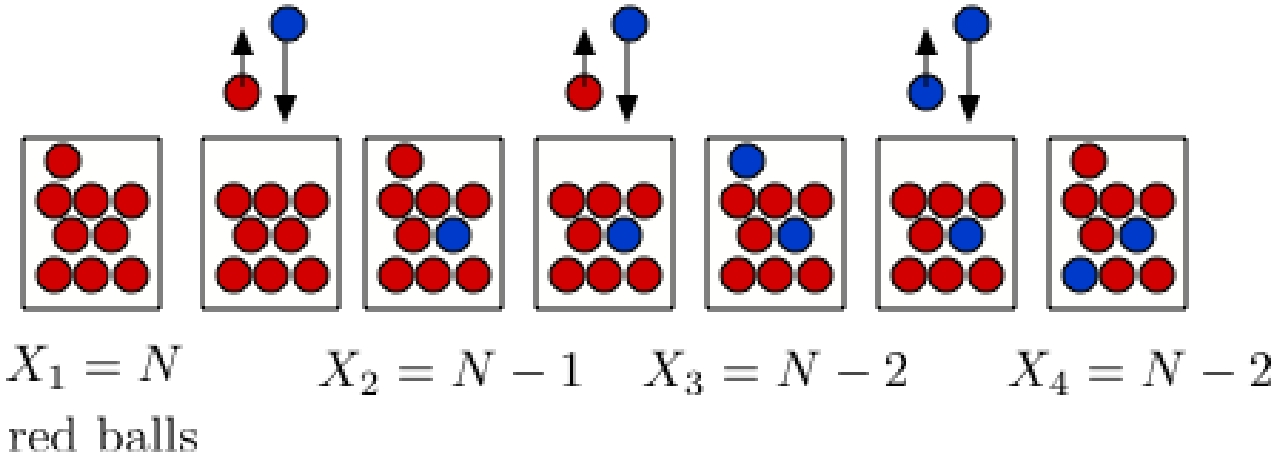

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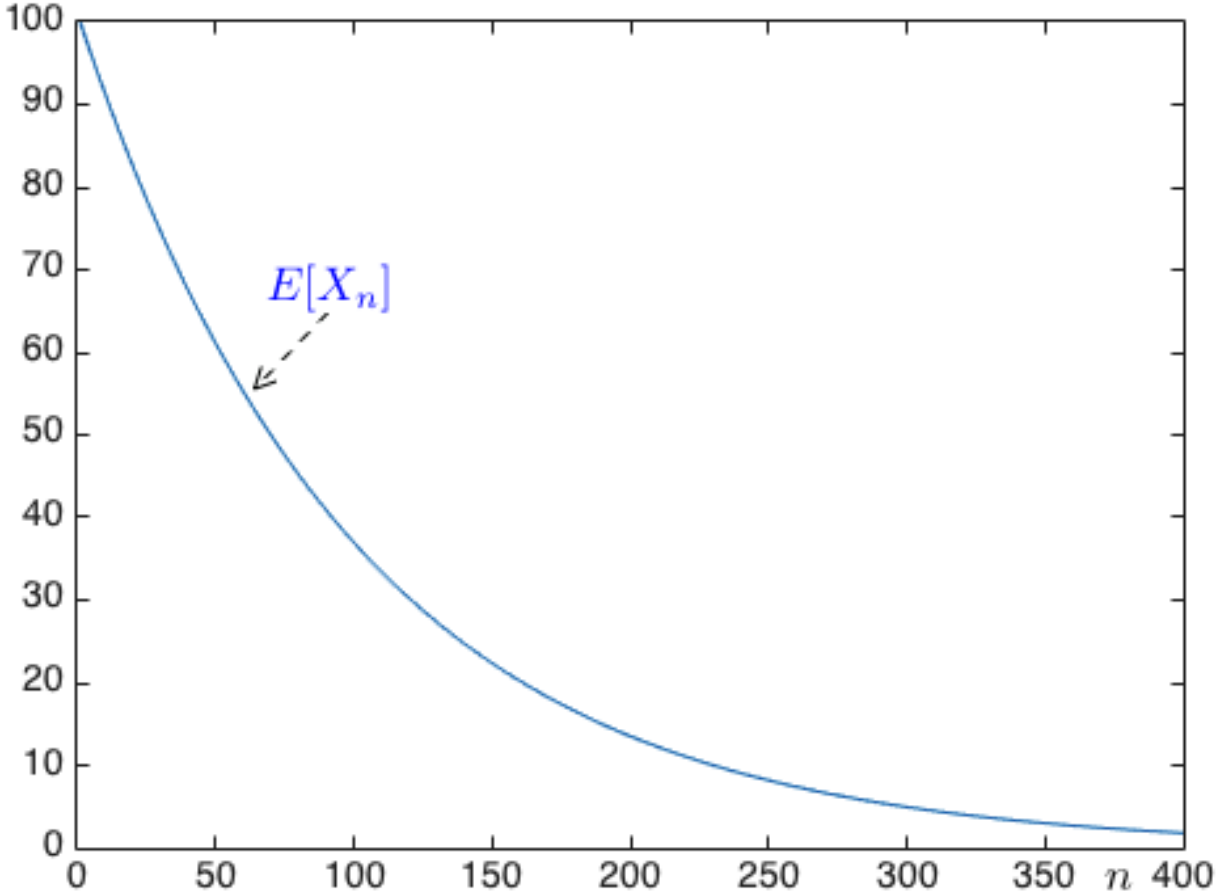

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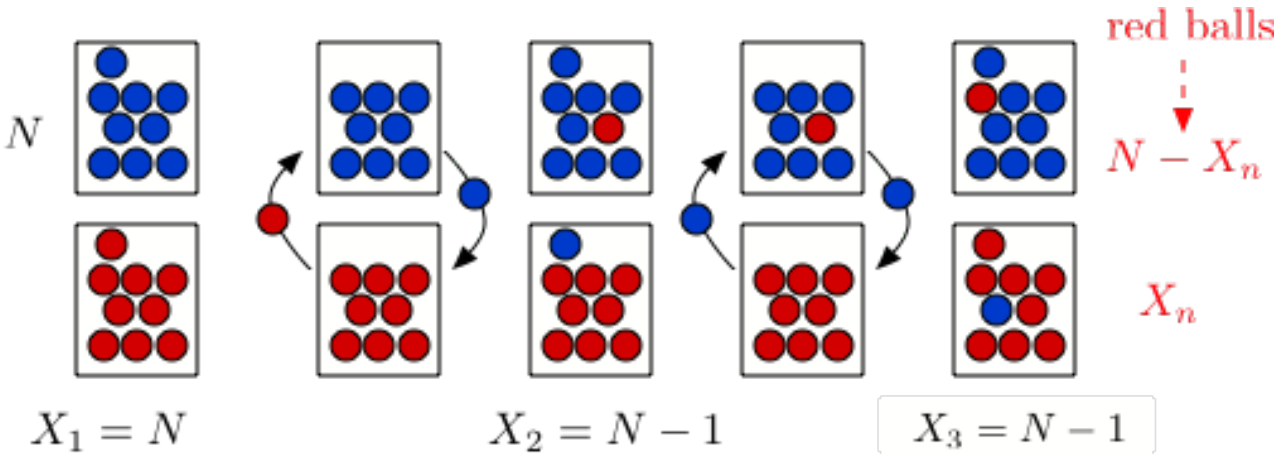
Application: Mixing



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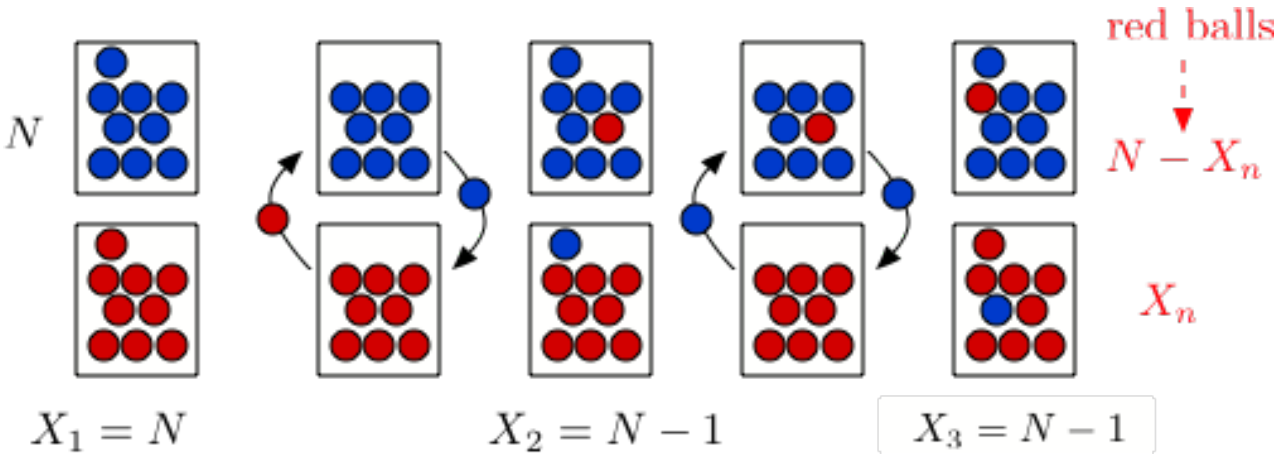

At each step, pick a ball from each well-mixed urn.

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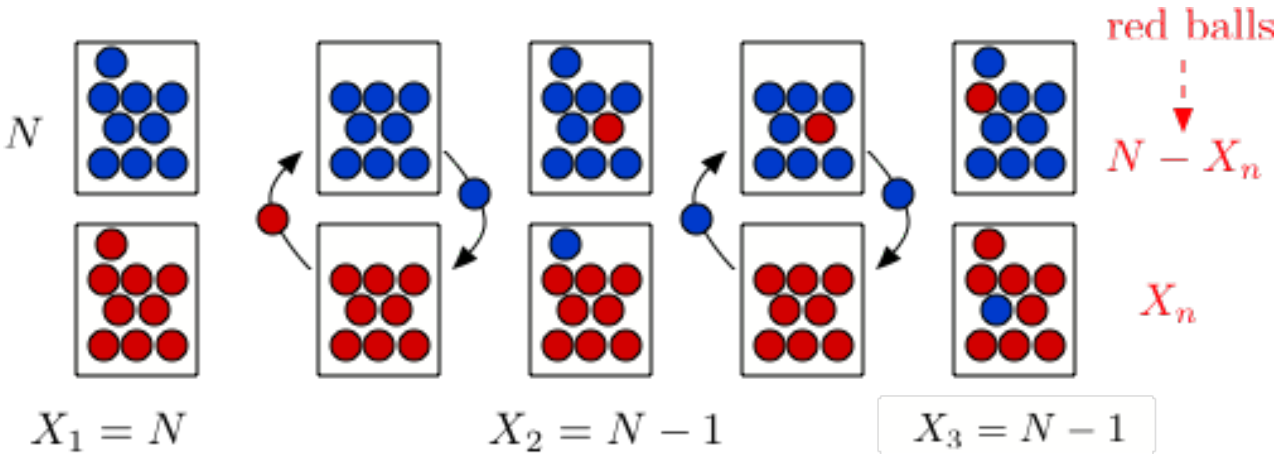

At each step, pick a ball from each well-mixed urn. We transfer them to the other urn.

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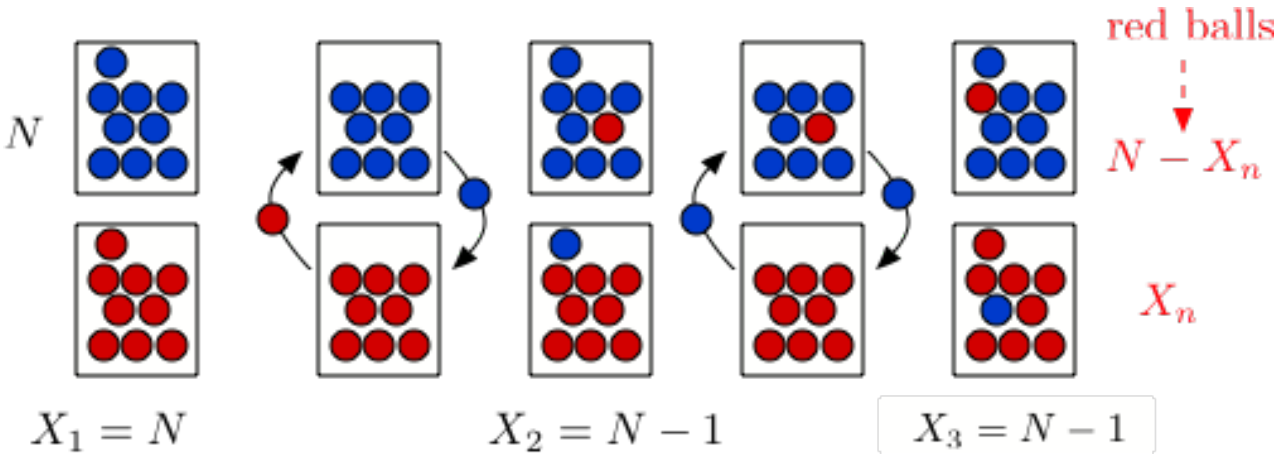

At each step, pick a ball from each well-mixed urn. We transfer them to the other urn. Let X_n be the number of red balls in the bottom urn at step n.

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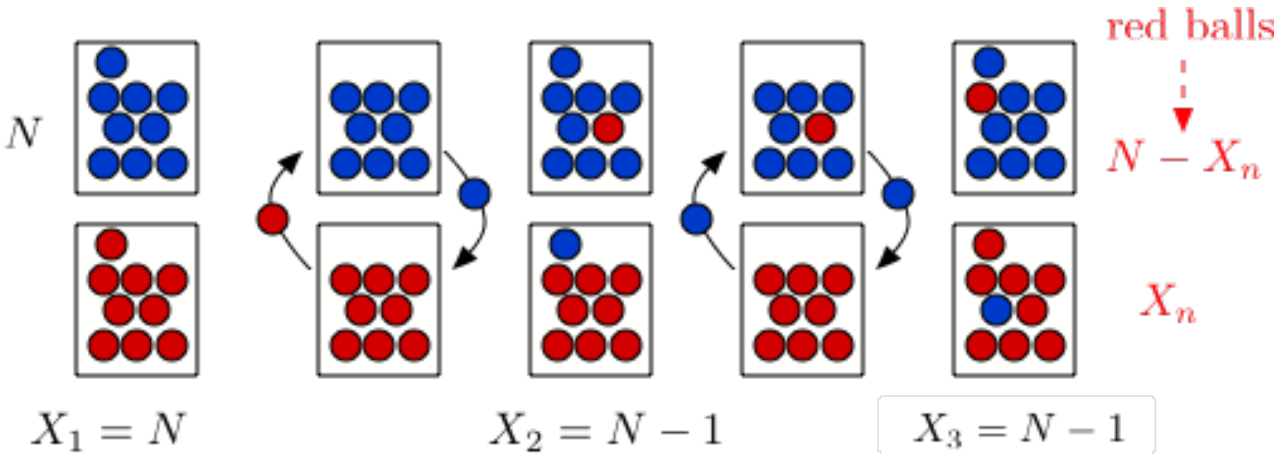

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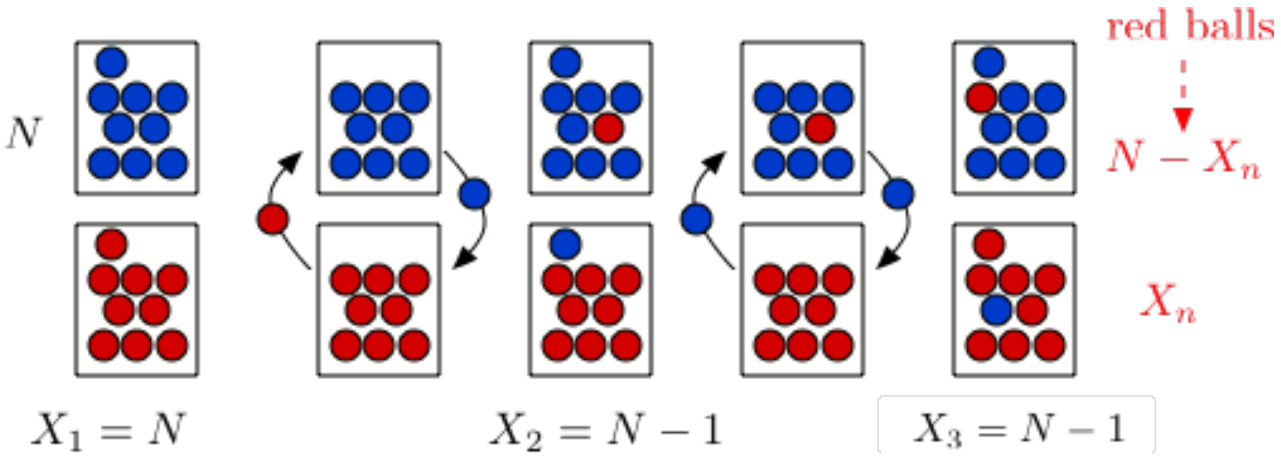
Application: Mixing



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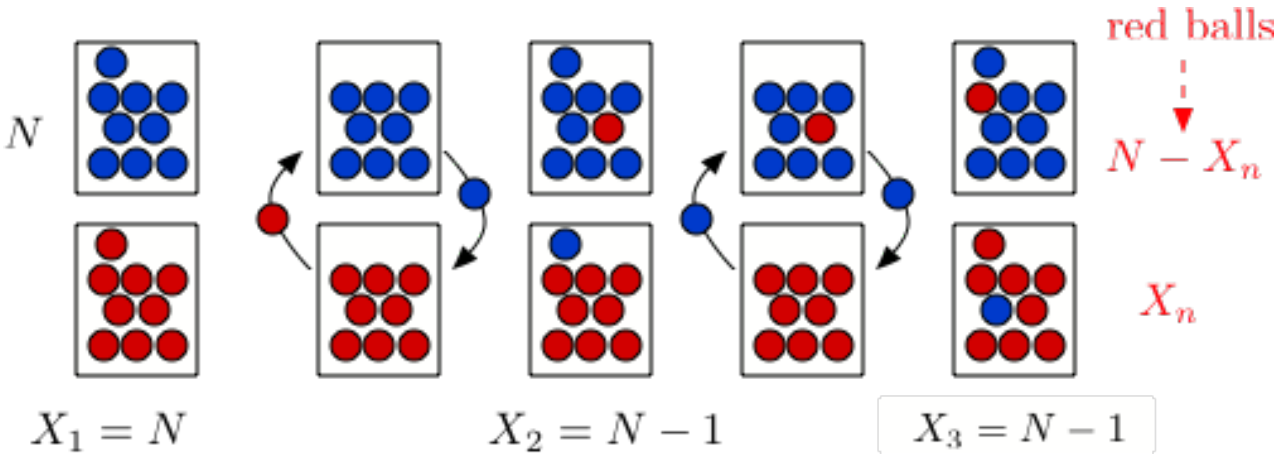

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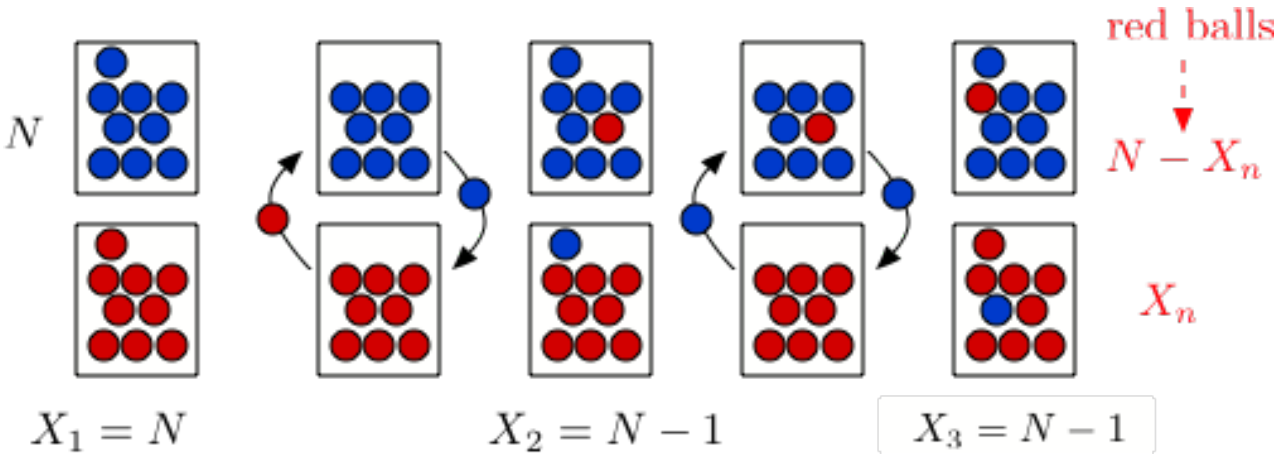

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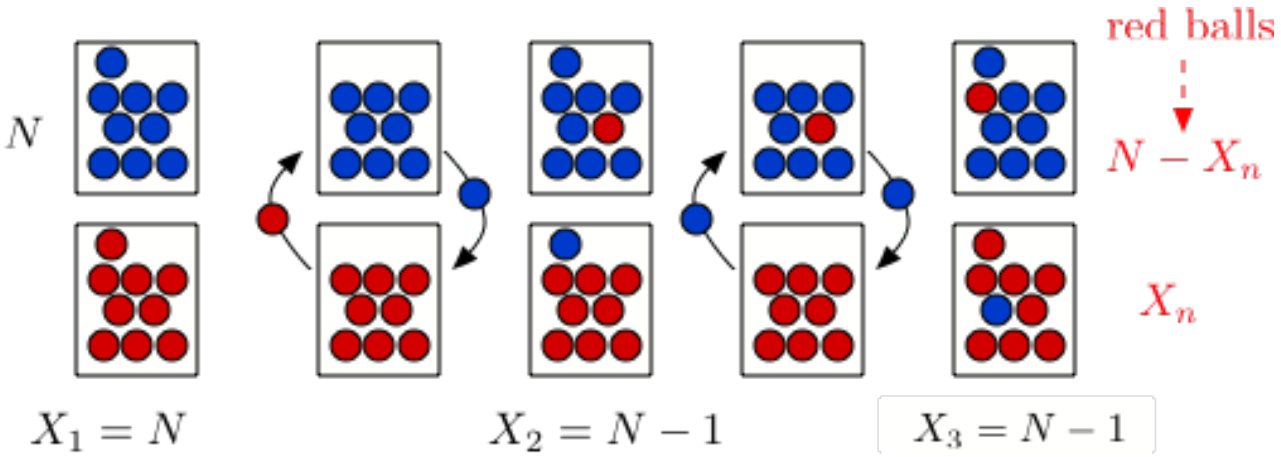

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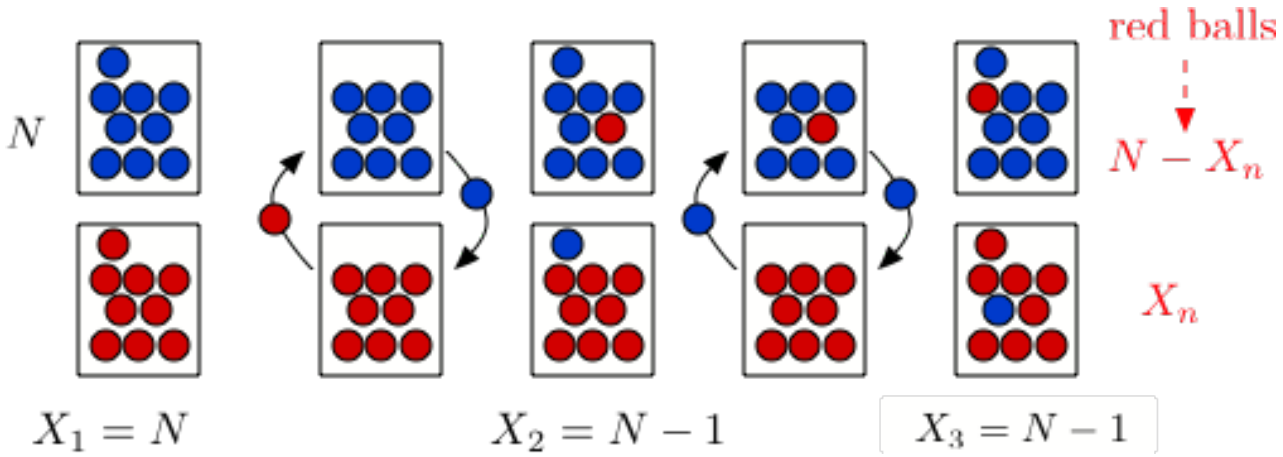

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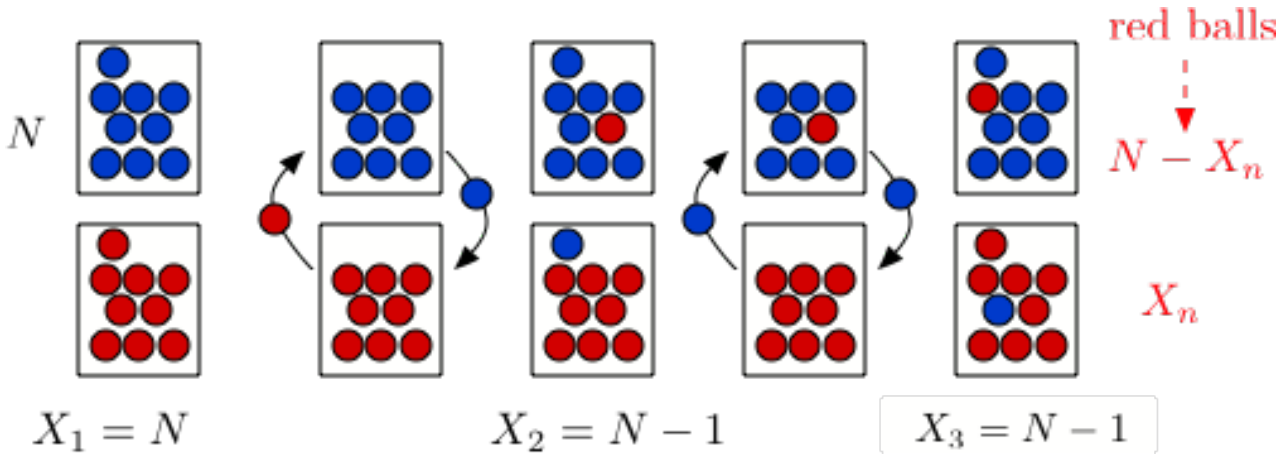

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Hence,

$$E[X_n] = \frac{1 - \rho^{n-1}}{1 - \rho} + \rho^{n-1} N, n \geq 1.$$

Application: Mixing

Here is the plot.

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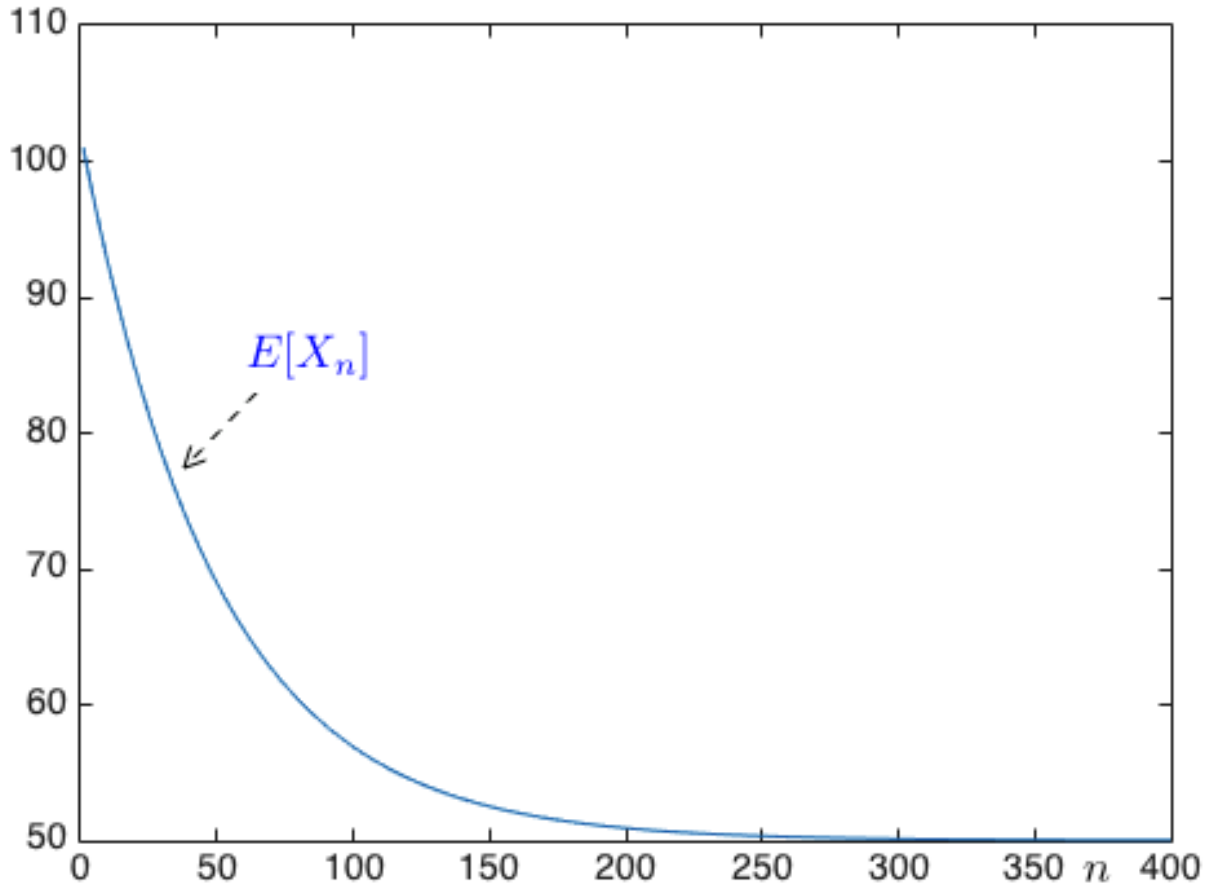

Application: Going Viral

Consider a social network (e.g., Twitter).

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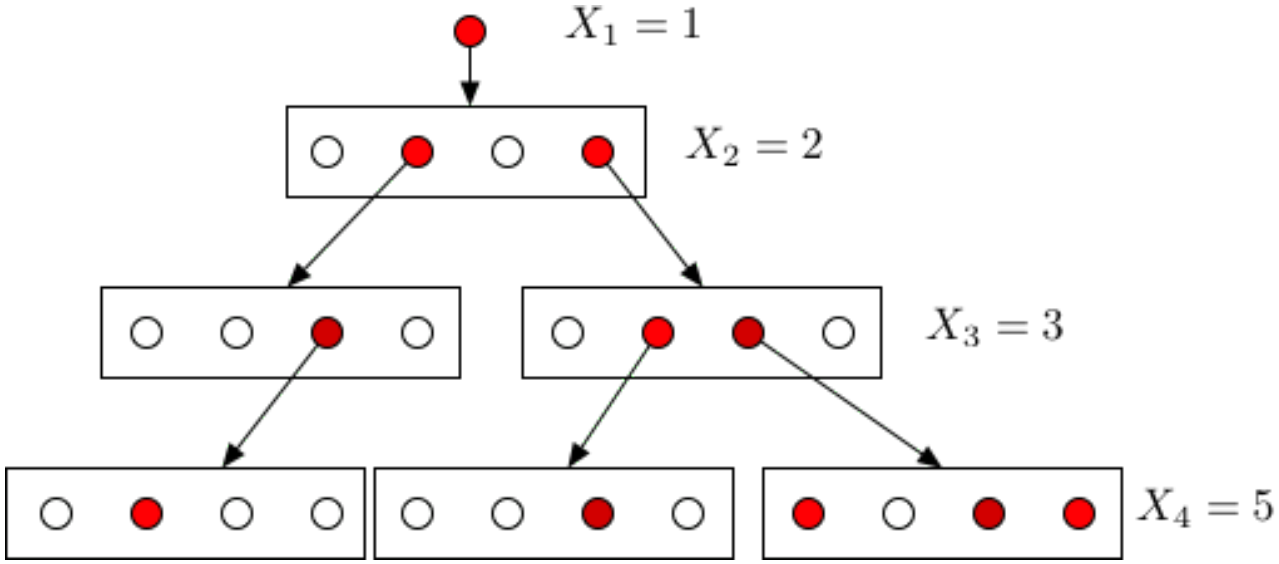

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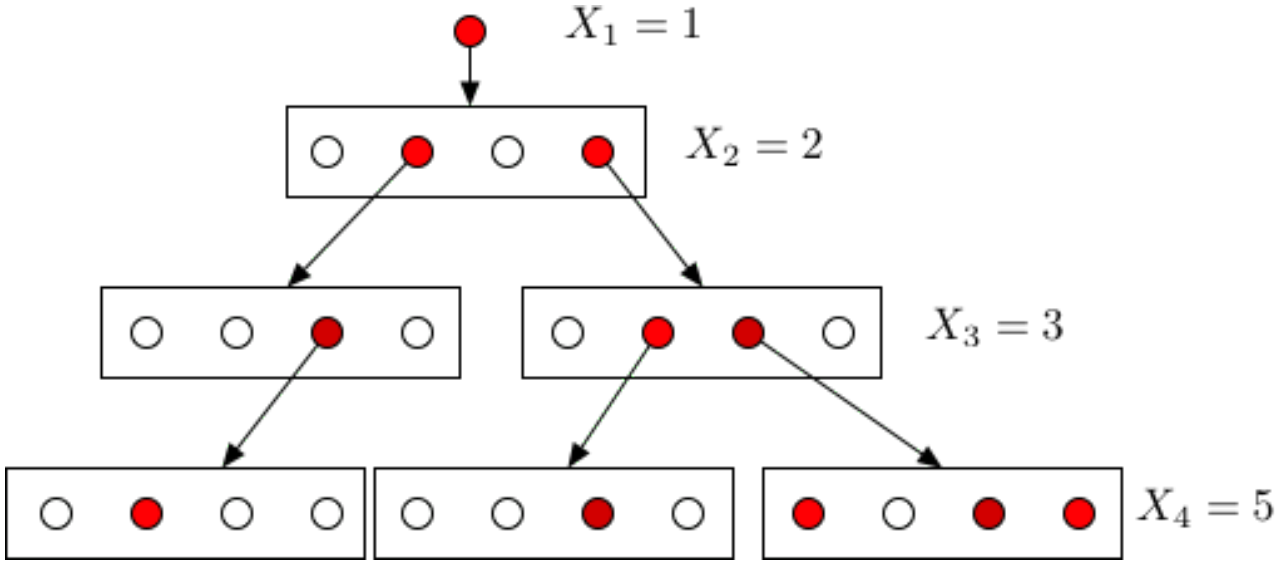
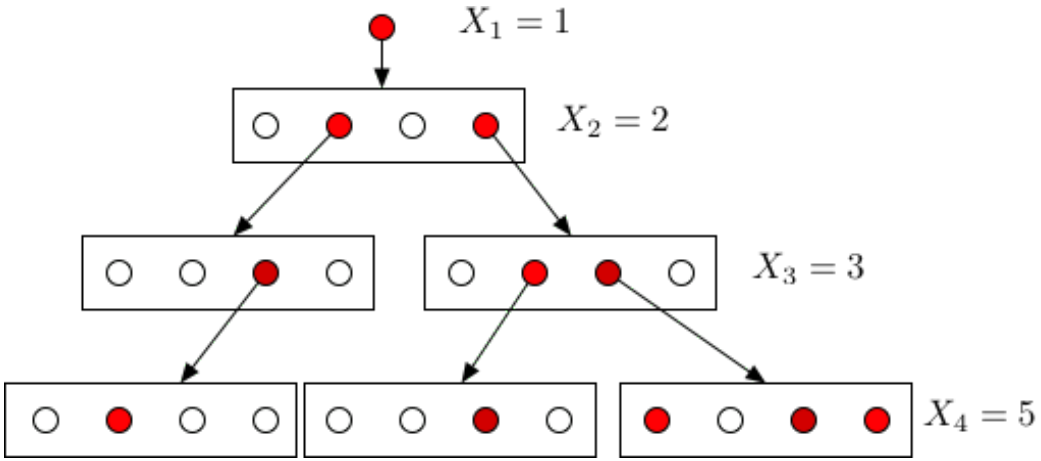


In this example, $d = 4$.

Application: Going Viral



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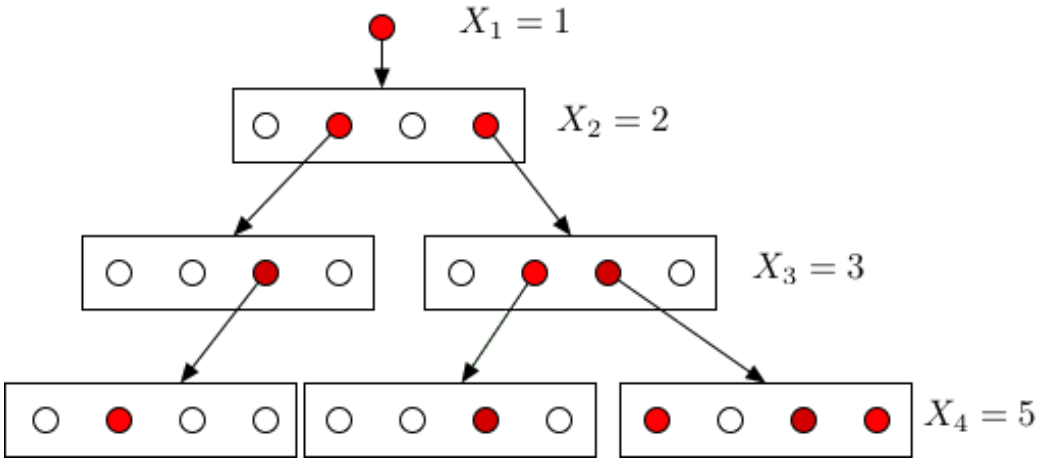
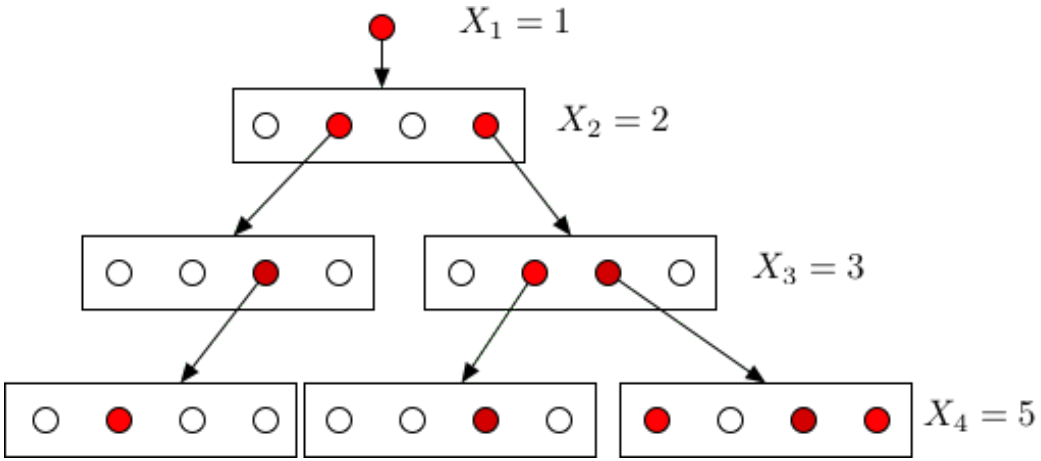


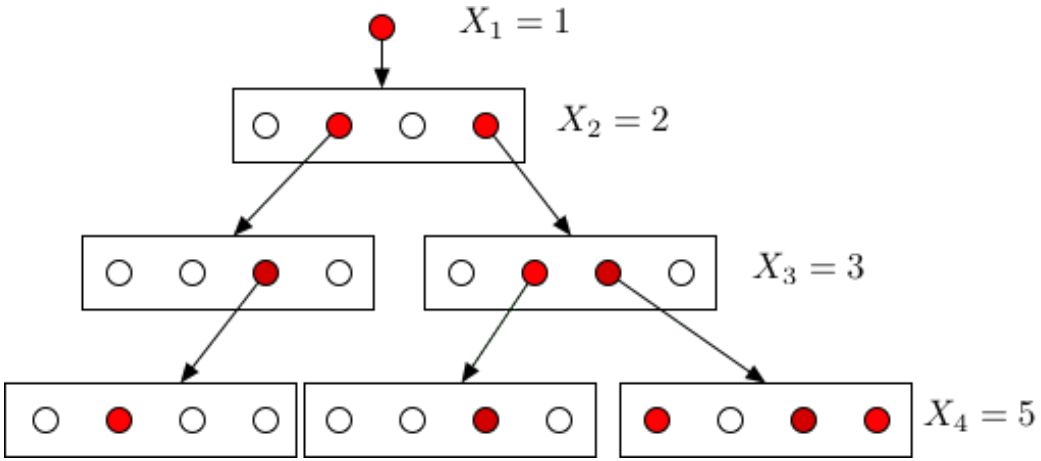
Fact:

Application: Going Viral



Fact: Let $X = \sum_{n=1}^{\infty} X_n$.

Application: Going Viral



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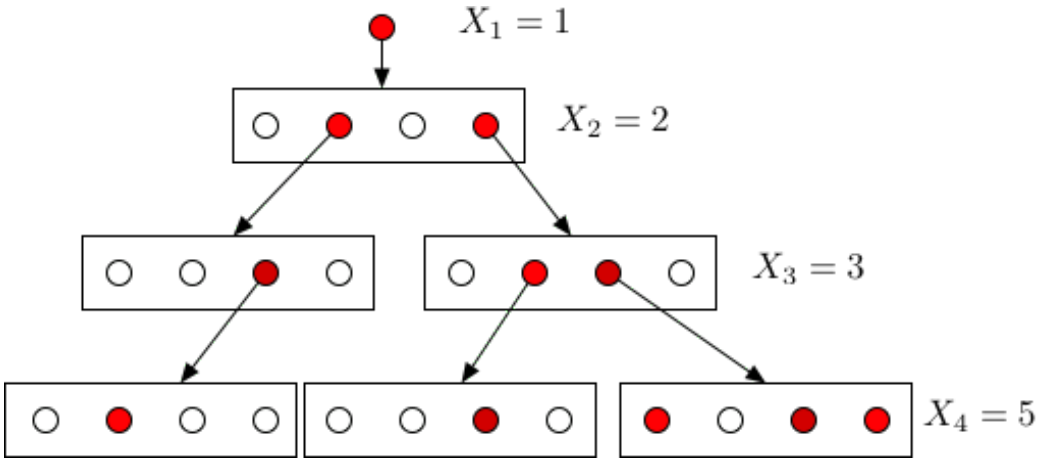
Application: Going Viral



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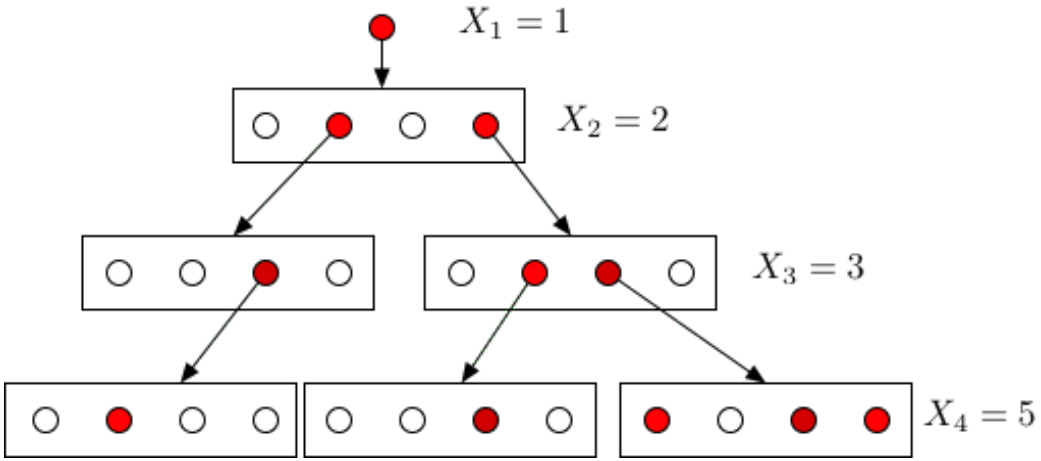
Application: Going Viral



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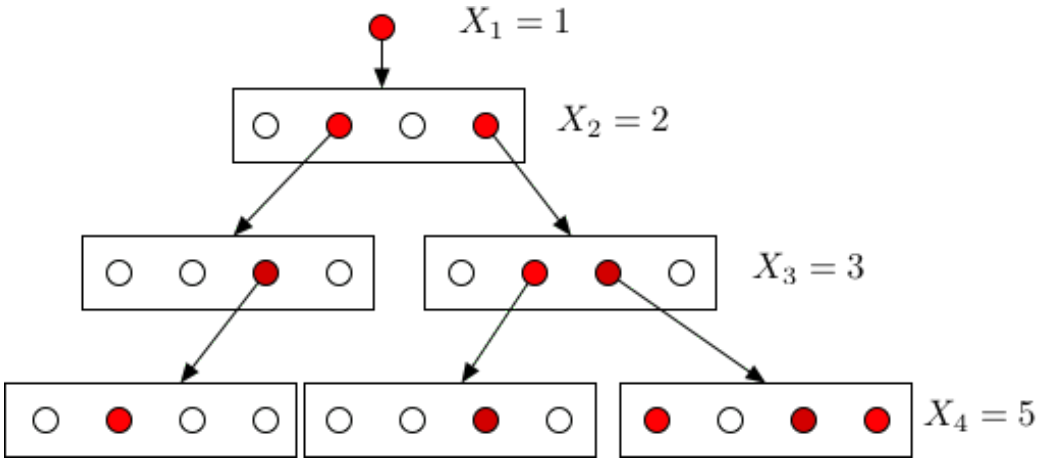
Application: Going Viral



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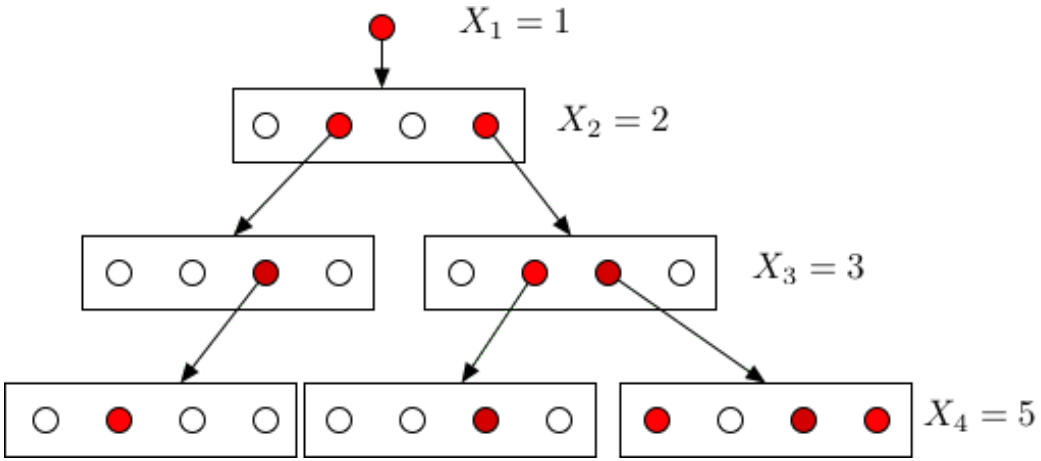
Application: Going Viral



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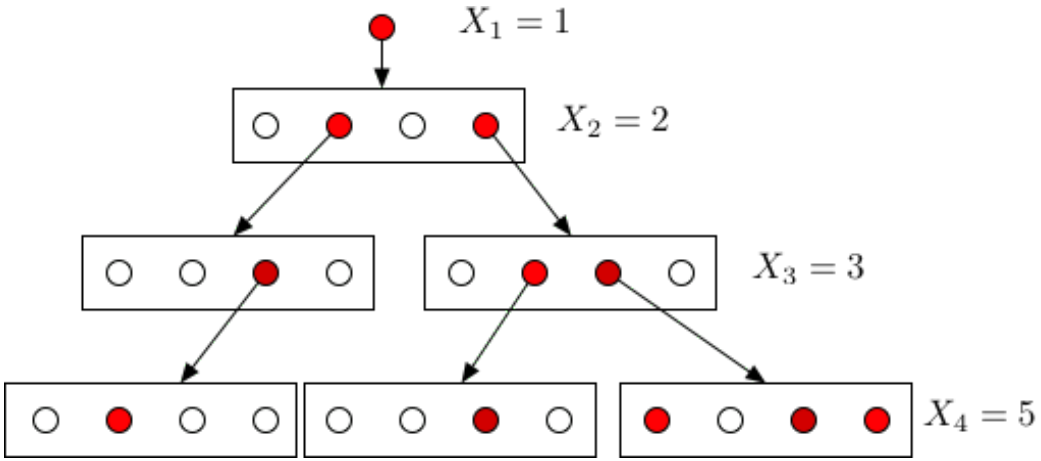

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If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1}$

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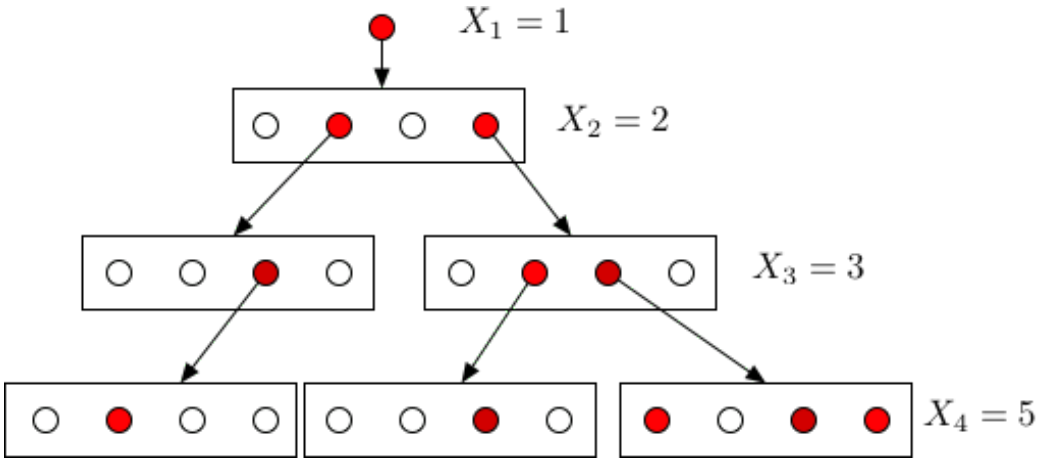

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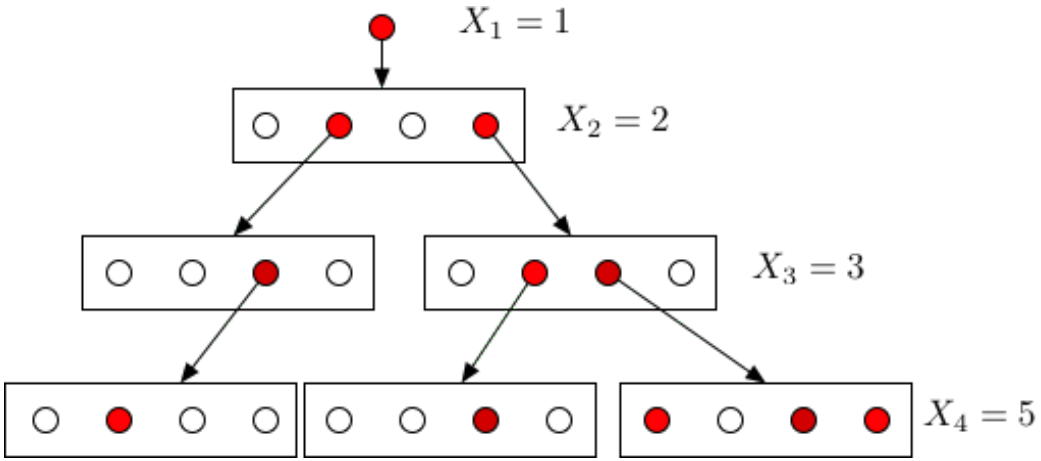

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If $pd \geq 1$, then for all C one can find n s.t.

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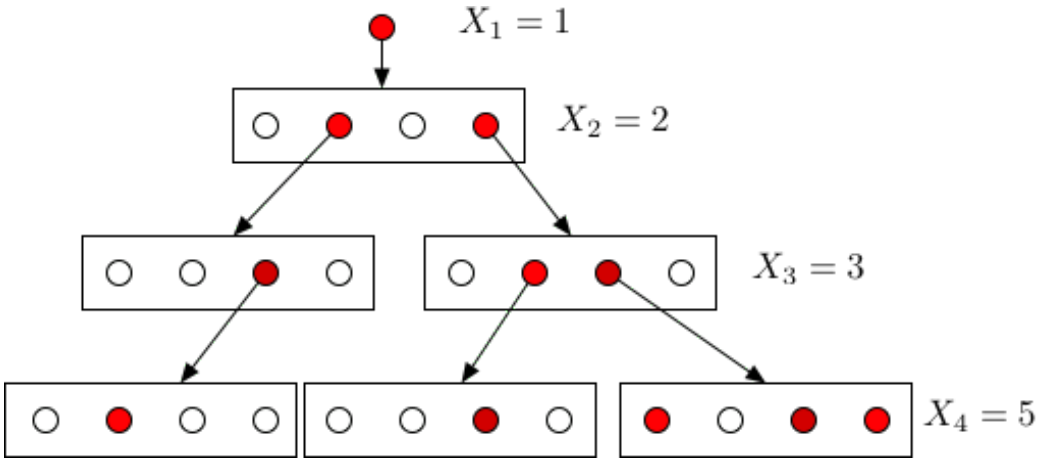
$\square$

Application: Going Viral



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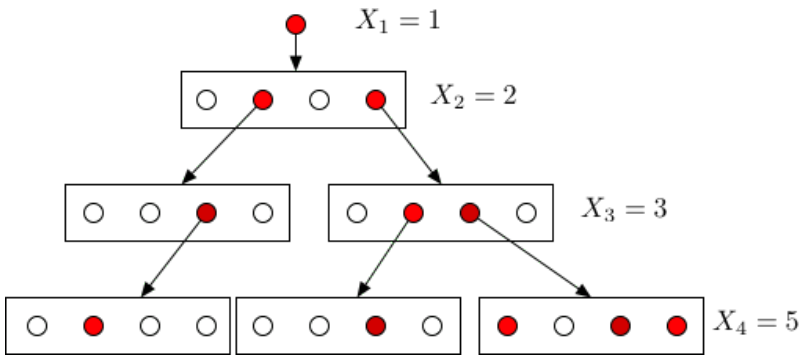
In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



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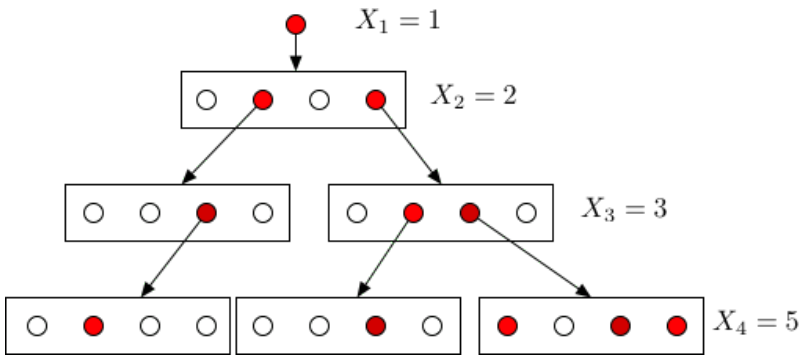

An easy extension:

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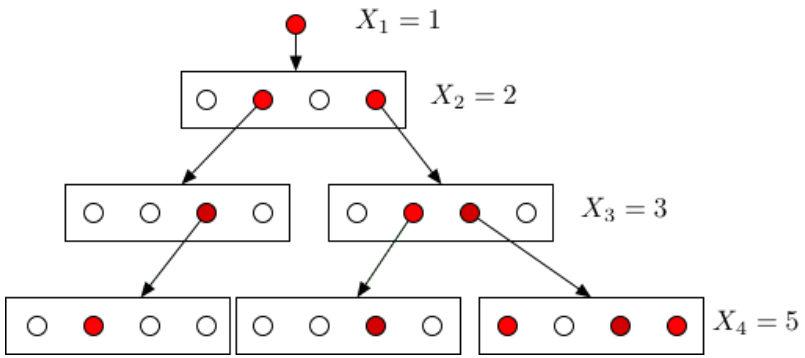
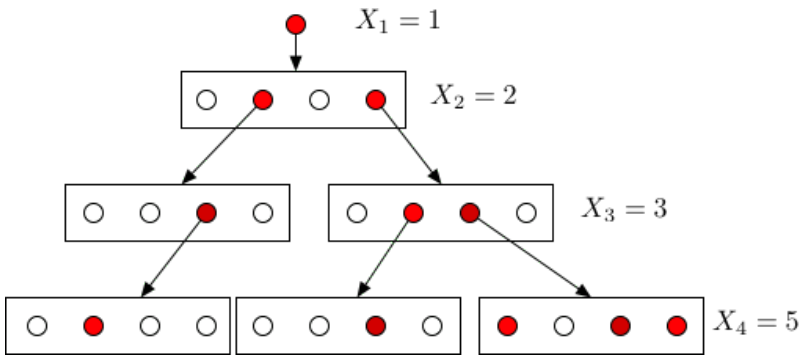


An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$.

Application: Going Viral



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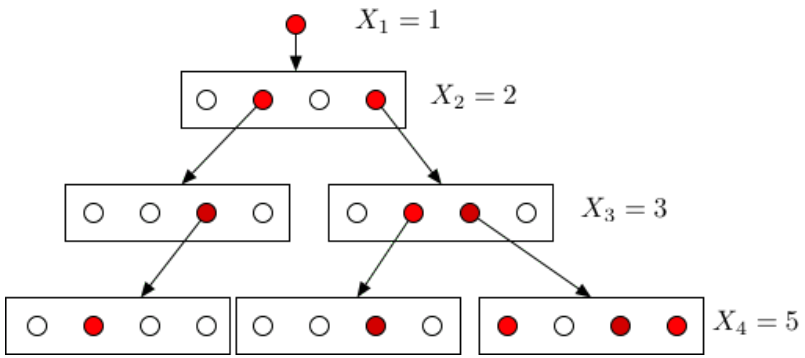

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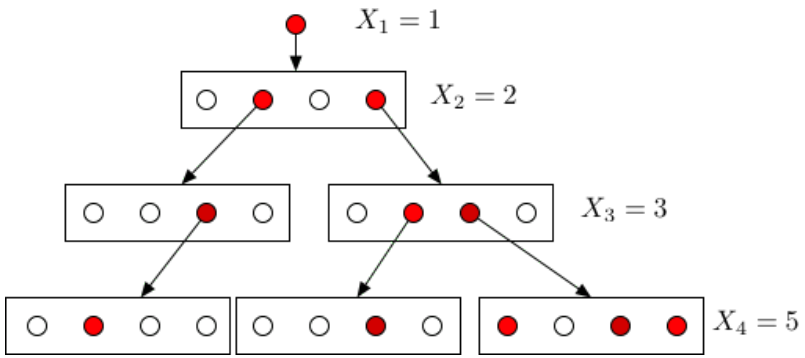

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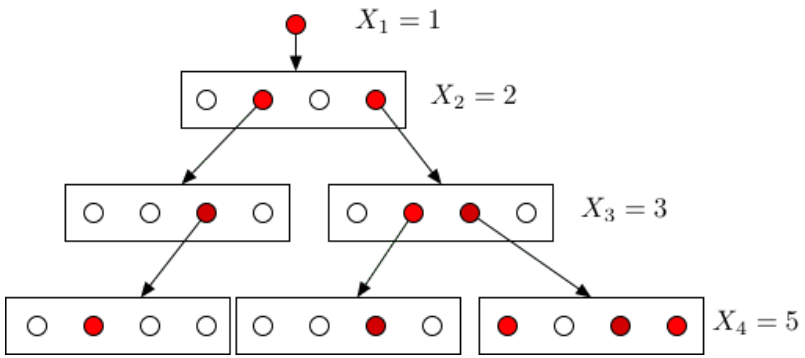

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$$E[X_{n+1} | X_n = k, D_1 = d_1, \ldots, D_k = d_k] = p(d_1 + \cdots + d_k).$$

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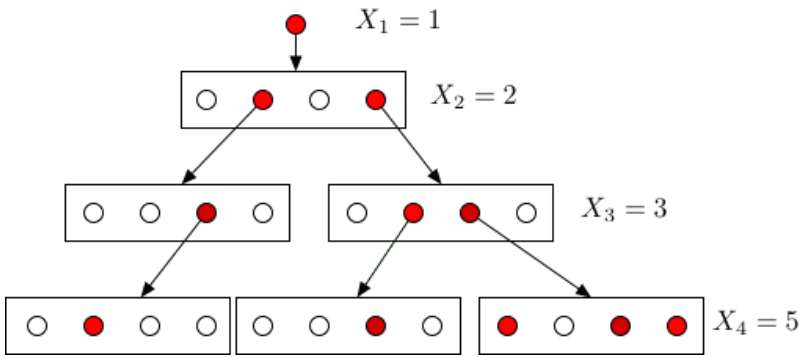

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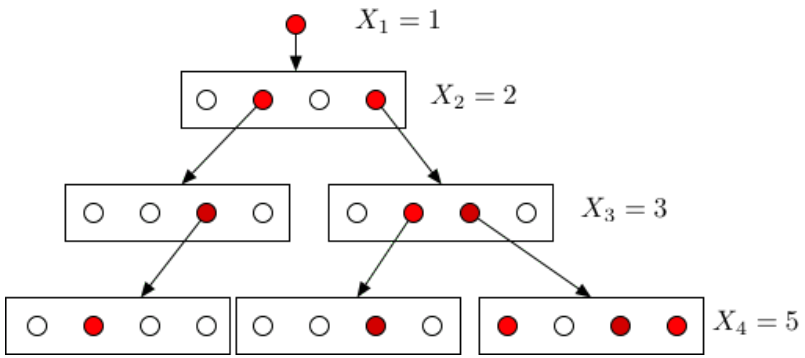

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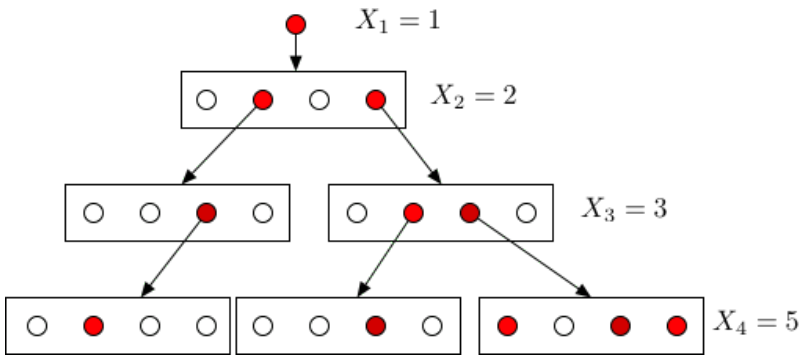

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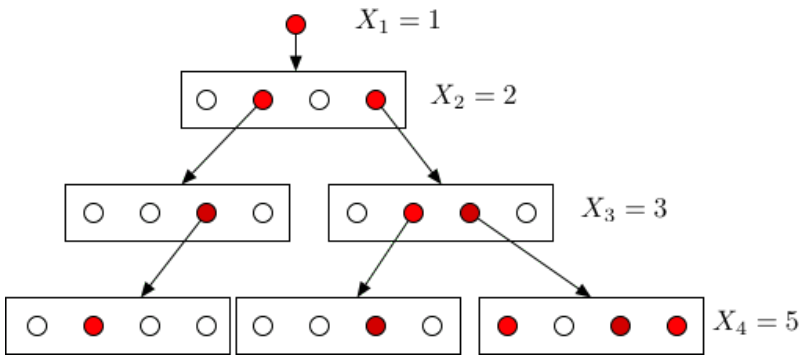

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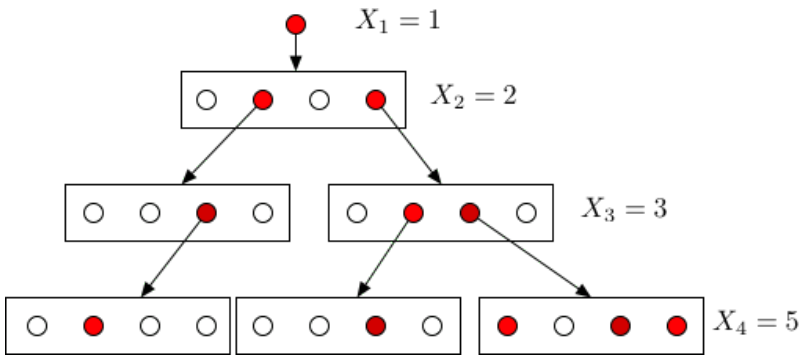

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We conclude as before.

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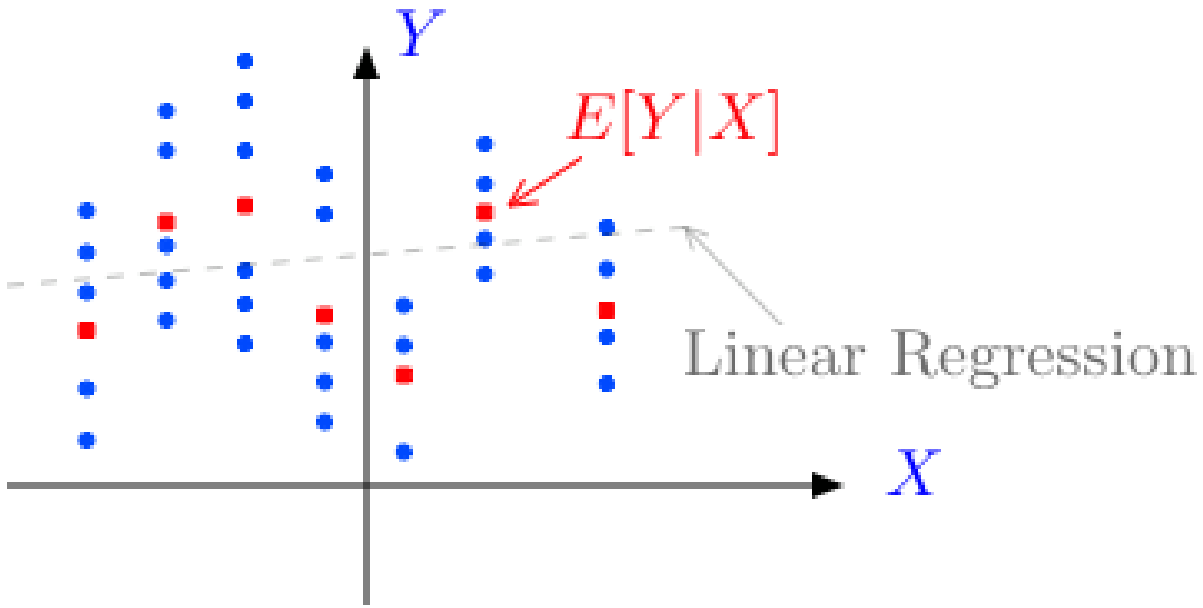

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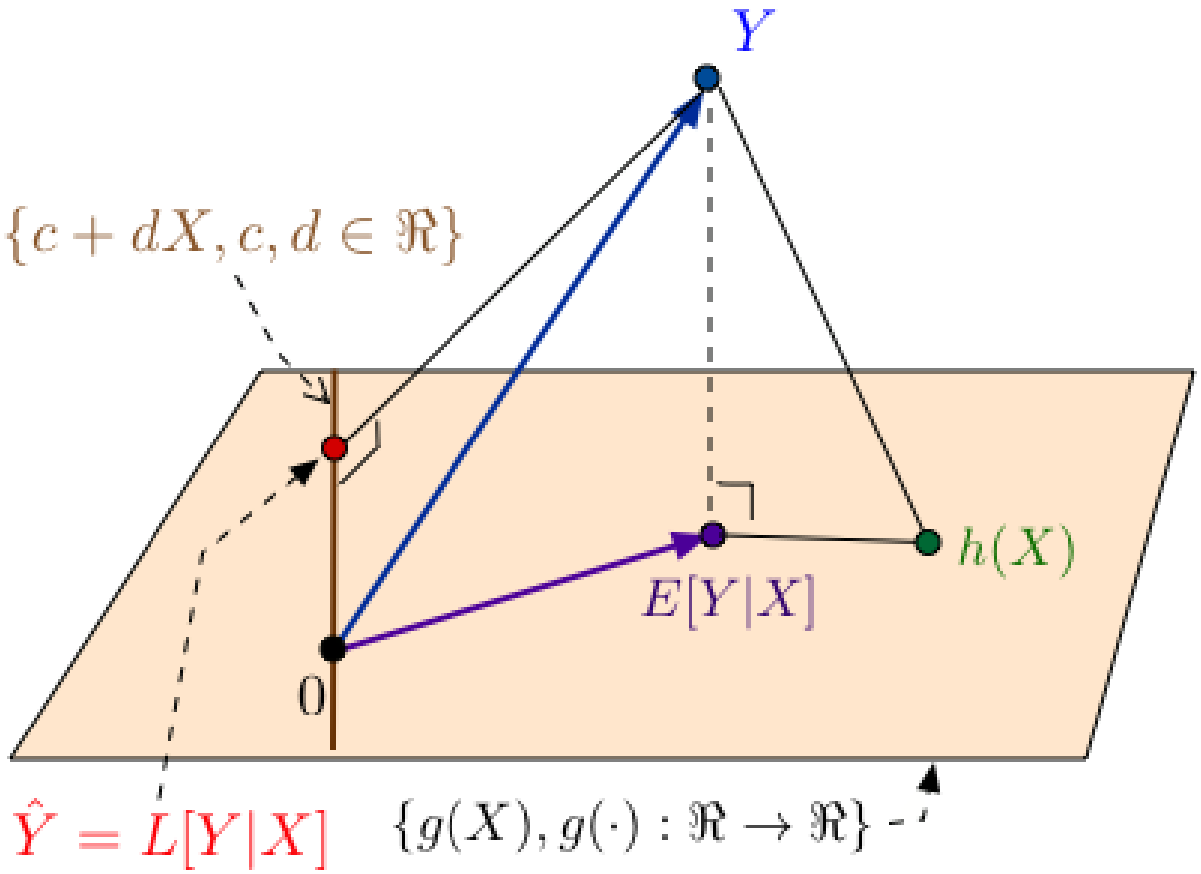

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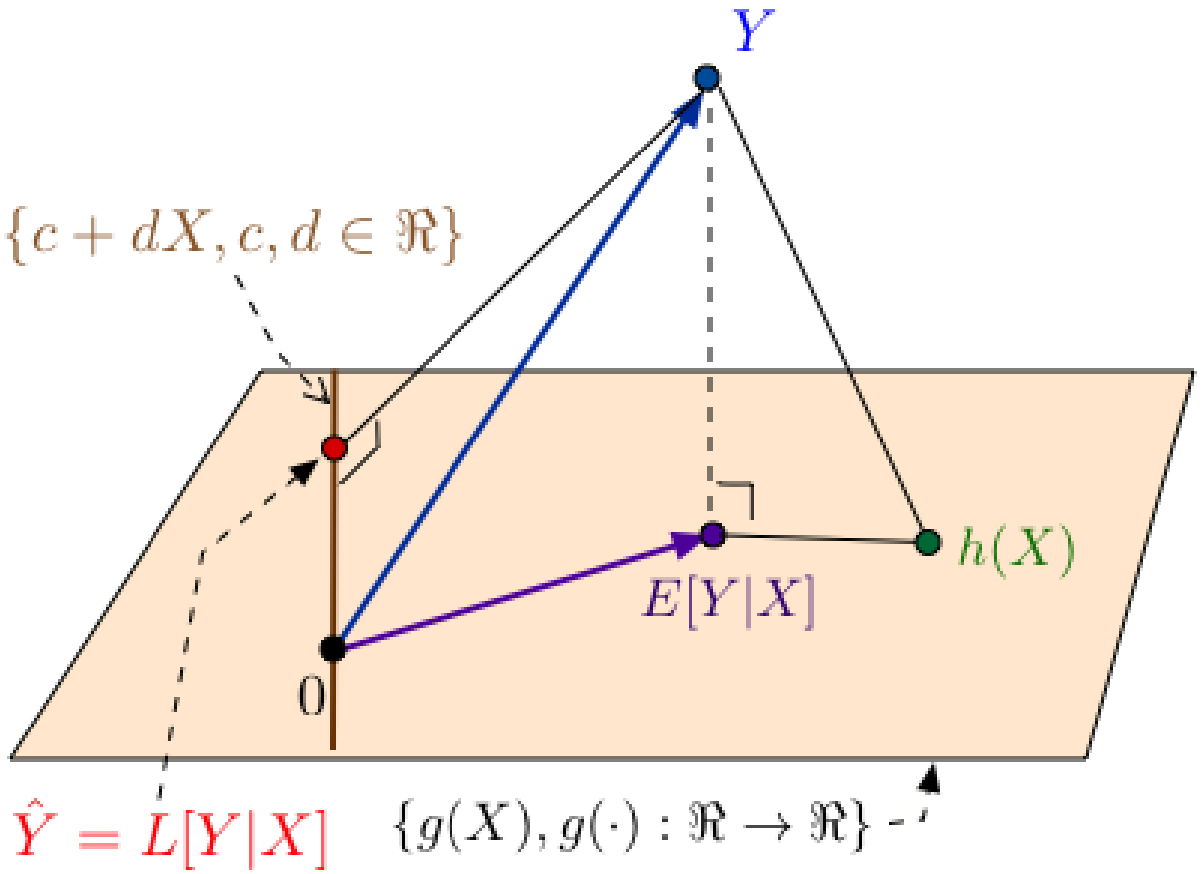

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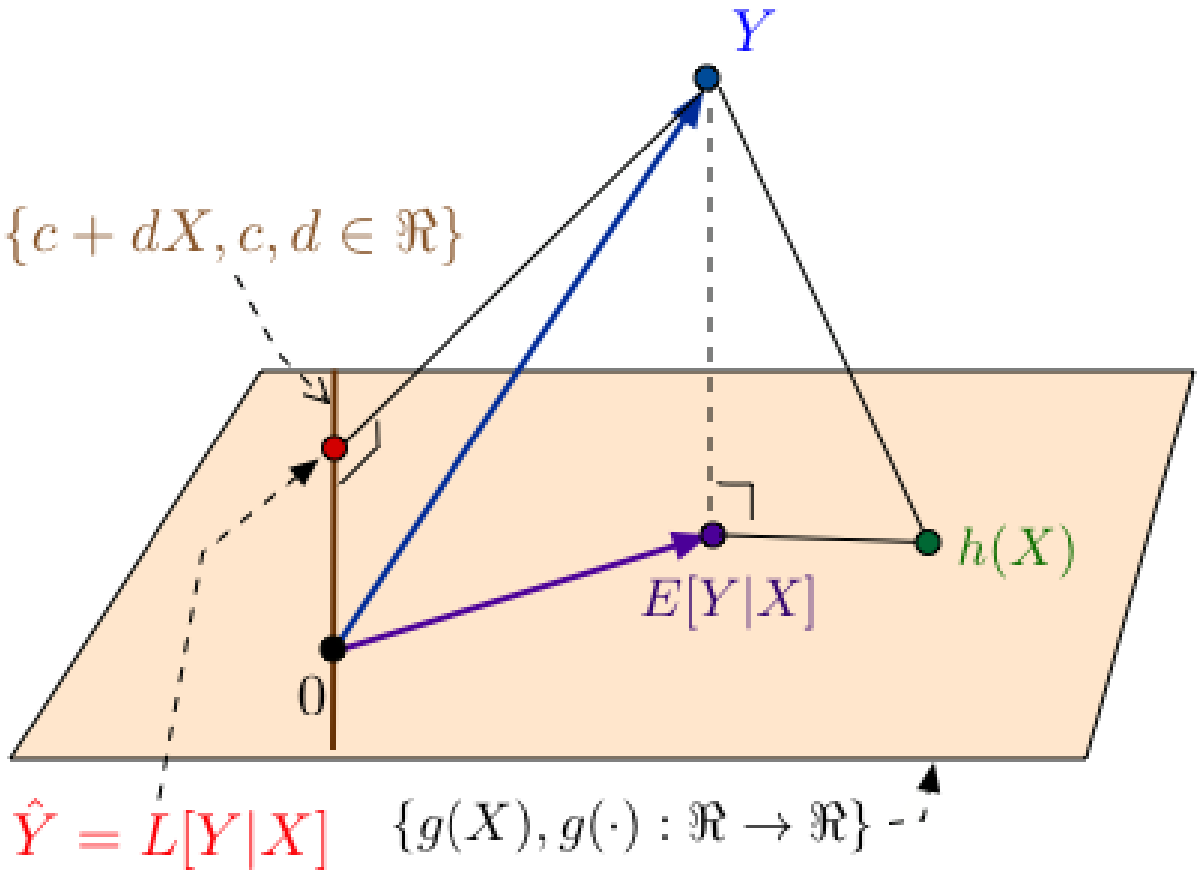

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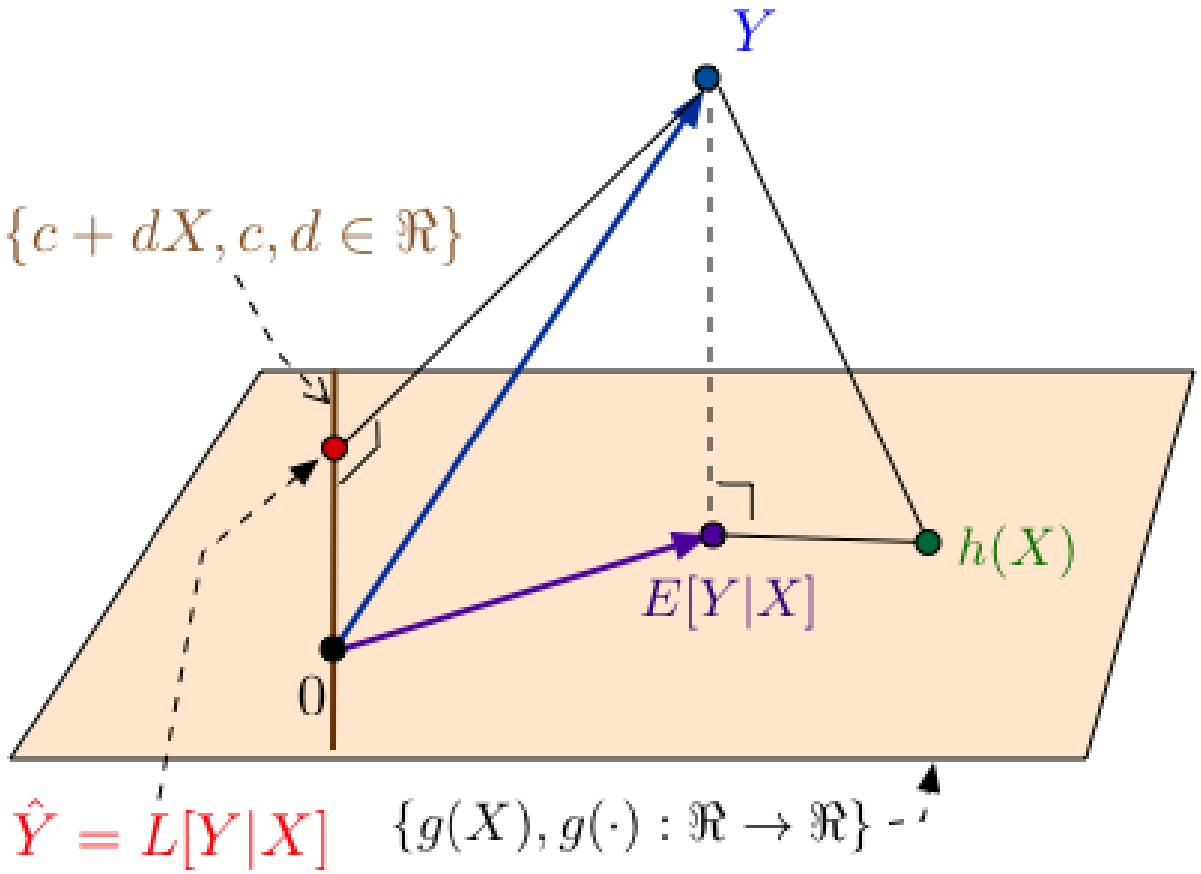

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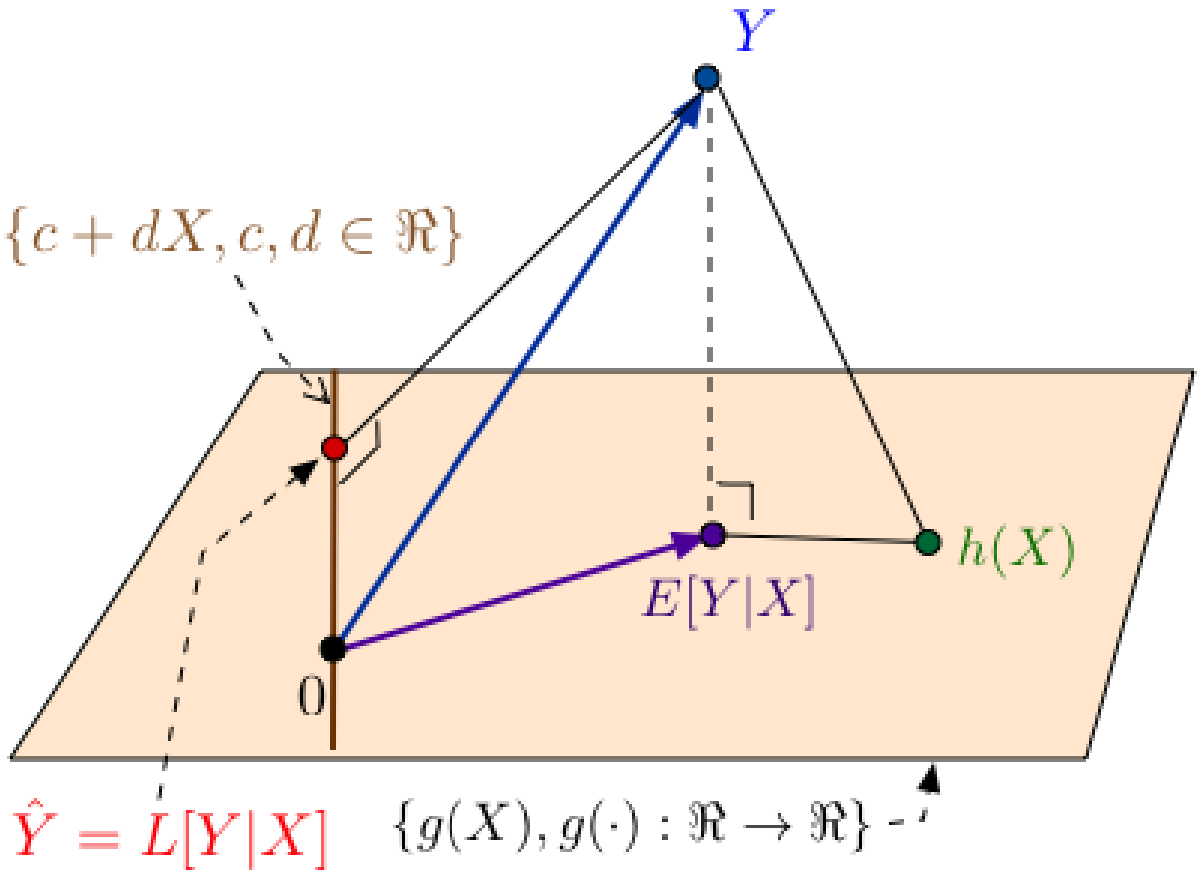

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