CS70: Jean Walrand: Lecture 35.

Conditional Expectation, Continuous Probability

Warning: This lecture is rated R.

1. Conditional Expectation
   - Review
   - Going Viral
   - Walt's Identity
   - CE = MMSE
2. Continuous Probability
   - Motivation.
   - Continuous Random Variables.
   - Cumulative Distribution Function.
   - Probability Density Function
   - Expectation and Variance

Conditional Expectation

Definition Let X and Y be RVs on Ω. The conditional expectation of Y given X is defined as

$$E[Y|X] = g(X)$$

where

$$g(x) := E[Y|X = x] := \sum_y y Pr[Y = y|X = x].$$

Properties of Conditional Expectation

$$E[Y|X=x] = \sum_y yPr[Y=y|X=x]$$

Theorem

(a) X, Y independent $\Rightarrow E[Y|X] = E[Y]$;

(b) $E[aY + bZ|X] = aE[Y|X] + bE[Z|X]$;

(c) $E[Yh(X)|X] = h(X)E[Y|X], \forall h(\cdot)$;

(d) $E[h(X)E[Y|X]] = E[h(X)Y], \forall h(\cdot)$;

(e) $E[E[Y|X]] = E[Y]$.

Application: Going Viral

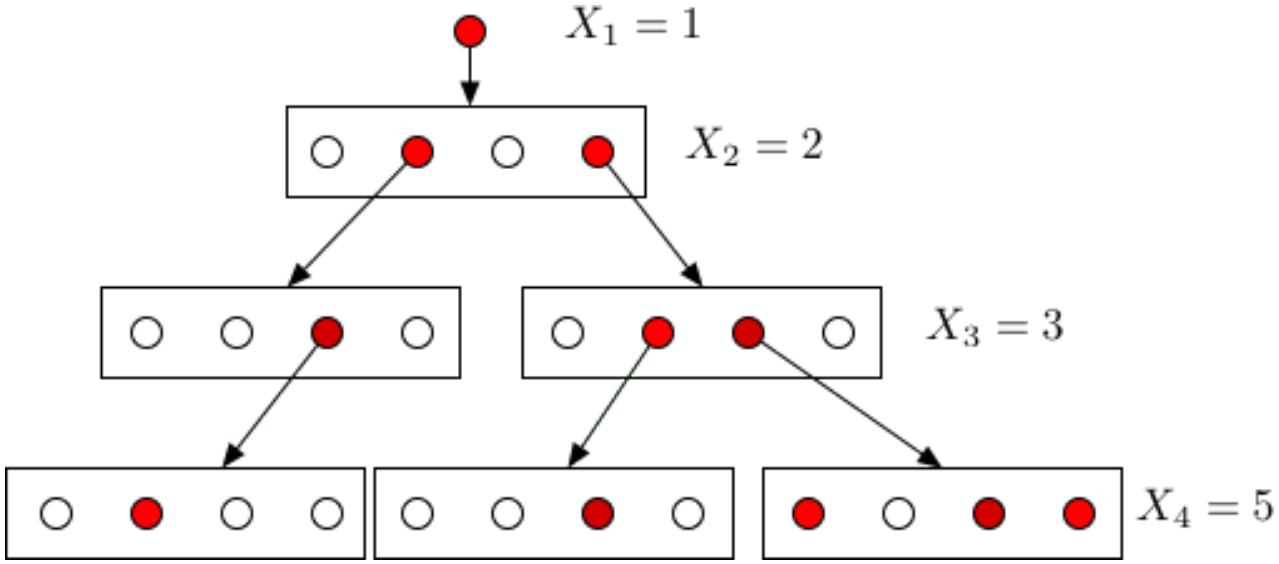
Consider a social network (e.g., Twitter).

You start a rumor (e.g., Walrand is really weird).

You have d friends. Each of your friend retweets w.p. p.

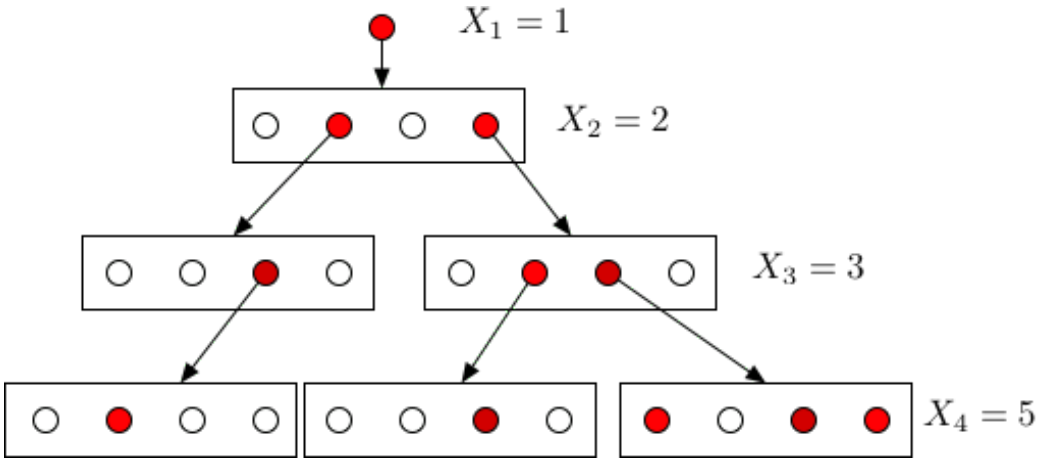
Each of your friends has d friends, etc.

Does the rumor spread? Does it die out (mercifully)?



In this example, $d = 4$.

Application: Going Viral



Fact: Let $X = \sum_{n=1}^{\infty} X_n$. Then, $E[X] < \infty$ iff $pd < 1$.

Proof:

Given $X_n = k$, $X_{n+1} = B(kd, p)$. Hence, $E[X_{n+1}|X_n = k] = kpd$.

Thus, $E[X_{n+1}|X_n] = pdX_n$. Consequently, $E[X_n] = (pd)^{n-1}, n \geq 1$.

If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1} \implies E[X] \leq (1 - pd)^{-1}$.

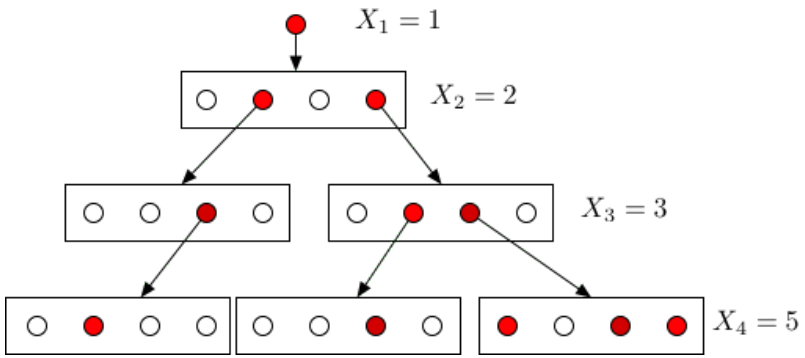
If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C.$$

$\square$

In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$. Then, the same fact holds.

To see this, note that given $X_n = k$, and given the numbers of friends $D_1 = d_1, \ldots, D_k = d_k$ of these X_n people, one has $X_{n+1} = B(d_1 + \cdots + d_k, p)$. Hence,

$$E[X_{n+1} | X_n = k, D_1 = d_1, \ldots, D_k = d_k] = p(d_1 + \cdots + d_k).$$

Thus, $E[X_{n+1} | X_n = k, D_1, \ldots, D_k] = p(D_1 + \cdots + D_k)$.

Consequently, $E[X_{n+1} | X_n = k] = E[p(D_1 + \cdots + D_k)] = pdk$.

Finally, $E[X_{n+1} | X_n] = pdX_n$, and $E[X_{n+1}] = pdE[X_n]$.

We conclude as before.

Application: Wald's Identity

Here is an extension of an identity we used in the last slide.

Theorem Wald's Identity

Assume that $X_1, X_2, \ldots$ and Z are independent, where

Z takes values in $\{0, 1, 2, \ldots\}$

and $E[X_n] = \mu$ for all $n \geq 1$.

Then,

$$E[X_1 + \cdots + X_Z] = \mu E[Z].$$

Proof:

$E[X_1 + \cdots + X_Z | Z = k] = \mu k.$

Thus, $E[X_1 + \cdots + X_Z | Z] = \mu Z.$

Hence, $E[X_1 + \cdots + X_Z] = E[\mu Z] = \mu E[Z].$ $\square$

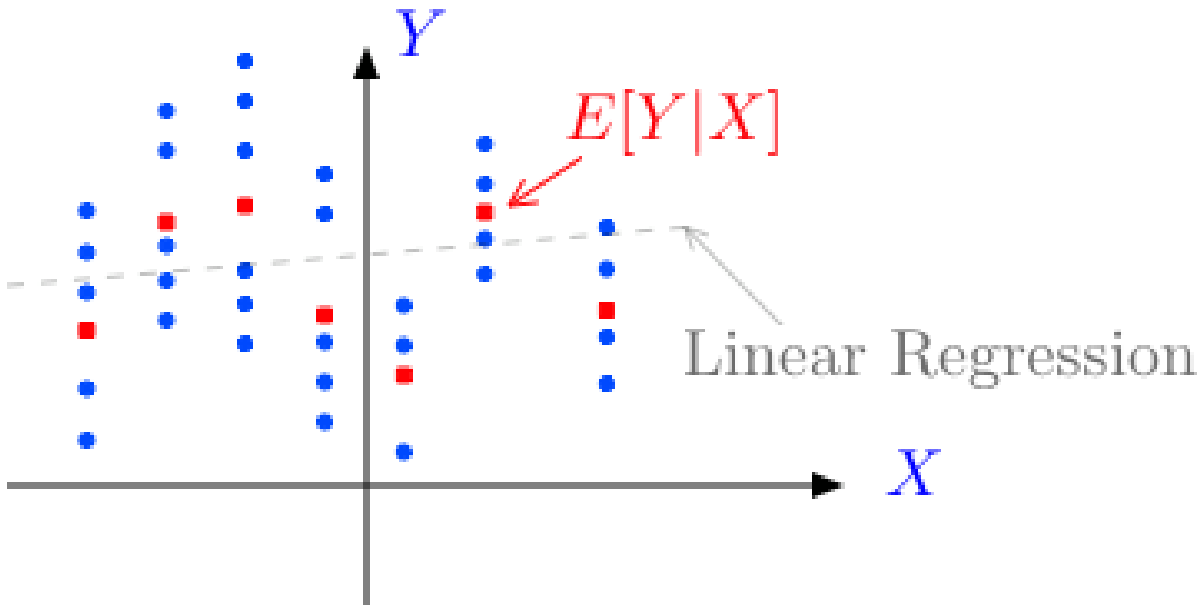
CE = MMSE

Theorem

$E[Y|X]$ is the 'best' guess about Y based on X.

Specifically, it is the function $g(X)$ of X that

$$\text{minimizes } E[(Y - g(X))^2].$$



CE = MMSE

Theorem CE = MMSE

$g(X) := E[Y|X]$ is the function of X that minimizes

$$E[(Y - g(X))^2]$$

.

Proof:

First recall the projection property of CE:

$$E[(Y - E[Y|X])h(X)] = 0, \forall h(\cdot).$$

That is, the error $Y - E[Y|X]$ is *orthogonal* to any $h(X)$.

CE = MMSE

Theorem CE = MMSE
$g(X) := E[Y|X]$ is the function of X that minimizes

$$E[(Y - g(X))^2]$$

.
Proof: Let $h(X)$ be any function of X. Then

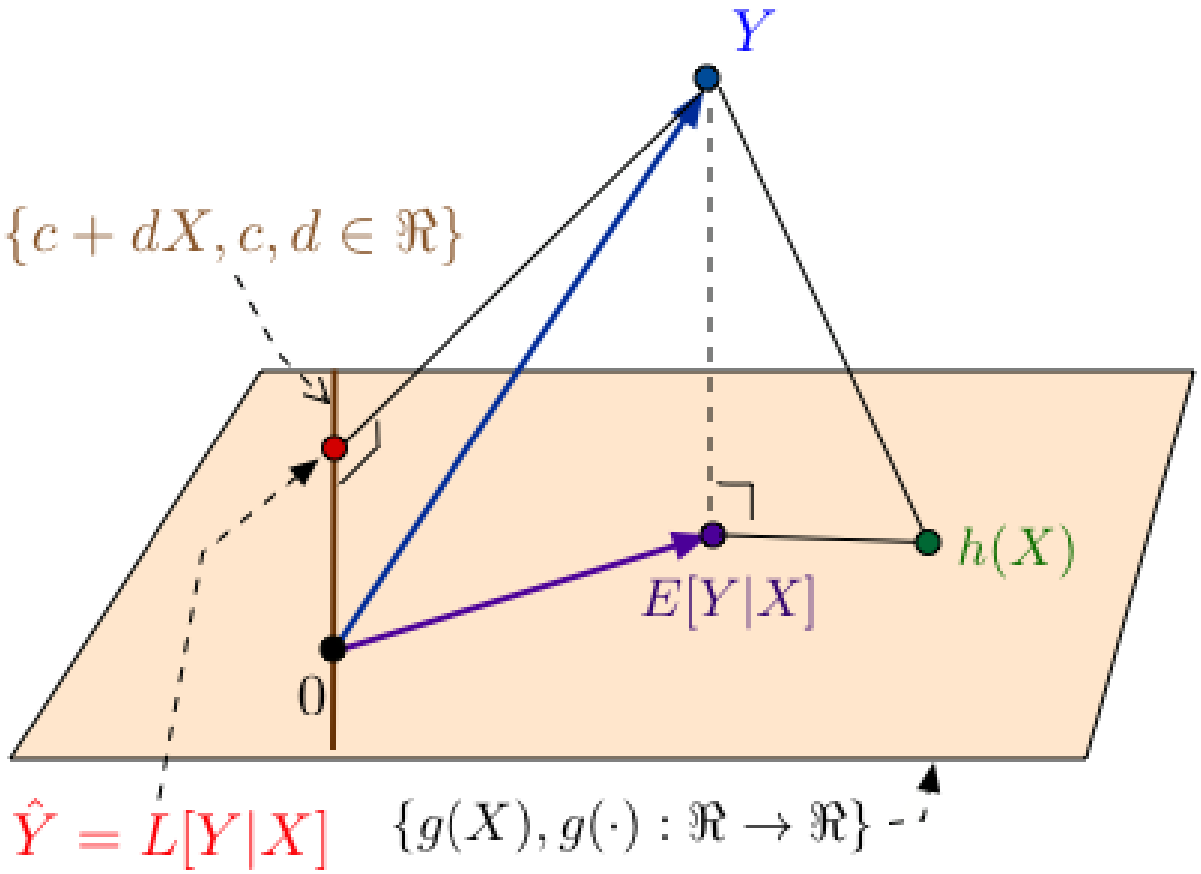
$$\begin{aligned} E[(Y - h(X))^2] &= E[(Y - g(X) + g(X) - h(X))^2] \\ &= E[(Y - g(X))^2] + E[(g(X) - h(X))^2] \\ &\qquad + 2E[(Y - g(X))(g(X) - h(X))]. \end{aligned}$$

But,

$$E[(Y - g(X))(g(X) - h(X))] = 0 \text{ by the projection property.}$$

Thus, $E[(Y - h(X))^2] \geq E[(Y - g(X))^2]$. $\square$

$E[Y|X]$ and $L[Y|X]$ as projections



$L[Y|X]$ is the projection of Y on $\{a + bX, a, b \in \Re\}$: LLSE

$E[Y|X]$ is the projection of Y on $\{g(X), g(\cdot) : \Re \to \Re\}$: MMSE.

Continuous Probability - James Bond.

- Escapes from SPECTRE sometime during $1,000$ mile flight.
- Uniformly likely to be at any point along path.

What is the chance he is at any point along the path?

Discrete Setting: Uniorm over $\Omega = \{1, \ldots, 1000\}$.

Continuous setting: probability at any point in $[0, 1000]$?
Probability at any one of an infinite number of points is ..
...uh ...0?

Continuous Probability: the interval!

Consider $[a,b] \subseteq [0,\ell]$ (for James, $\ell = 1000$.)
Let $[a,b]$ also denote the **event** that point is in the interval $[a,b]$.

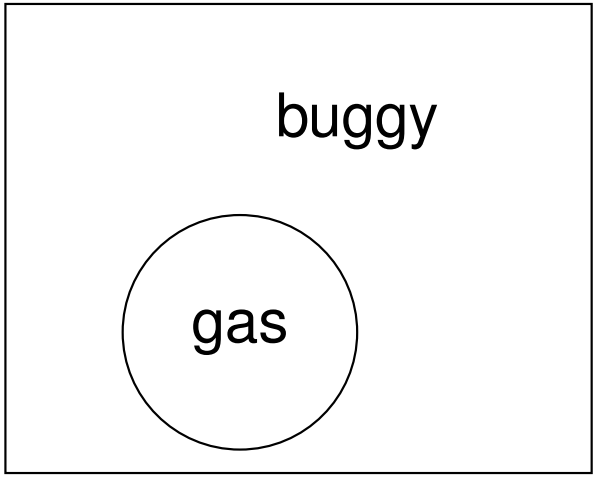
$$Pr[[a,b]] = \frac{\text{length of } [a,b]}{\text{length of } [0,\ell]} = \frac{b-a}{\ell} = \frac{b-a}{1000}.$$

Again, $[a,b] \subseteq \Omega = [0,\ell]$ are **events.**
Events in this space are unions of intervals.
Example: the event A - "within 50 miles of base" is
$[0,50] \cup [950,1000]$.

$$Pr[A] = Pr[[0,50]] + Pr[[950,10000]] = \frac{1}{10}.$$

Shooting..

Another Bond example:
Spectre is chasing him in a buggie.
Bond shoots at buggy and hits it at random spot.
What is the chance he hits gas tank?
Gas tank is a one foot circle and the buggy is 4×5 rectangle.



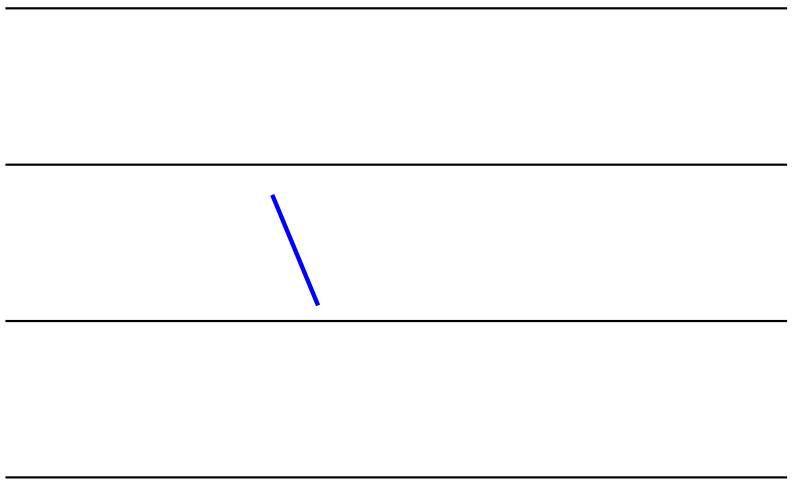
$\Omega = \{(x, y) : x \in [0, 4], y \in [0, 5]\}$.

The size of the event is $\pi(1)^2 = \pi$.
The "size" of the sample space which is 4×5.

Since uniform, probability of event is $\frac{\pi}{20}$.

Buffon's needle.

Throw a needle on a board with horizontal lines at random.

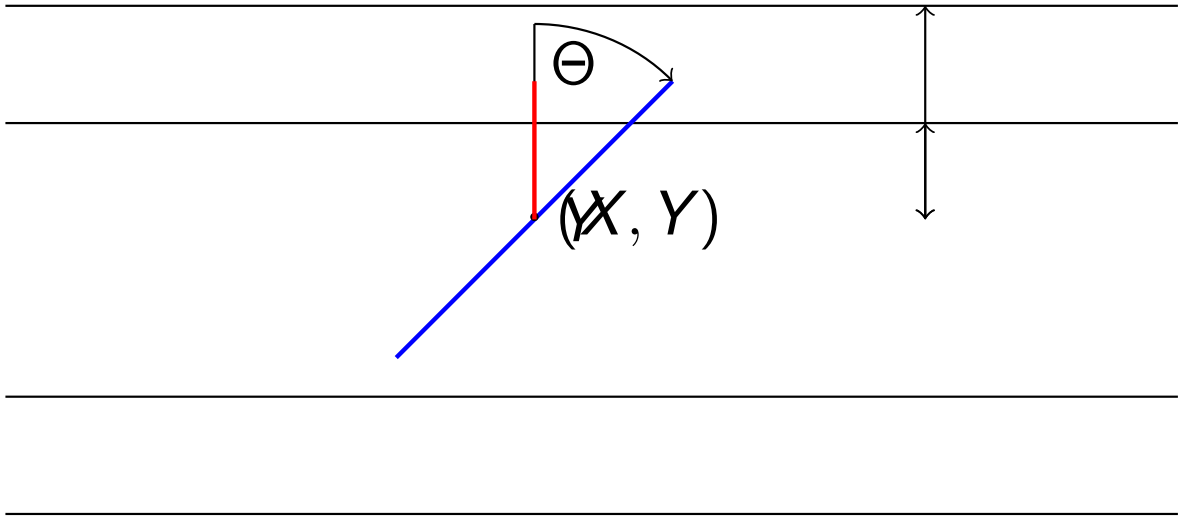


Lines 1 unit apart, needle has length 1.
What is the probability that the needle hits a line?
Clearly...

$$\frac{2}{\pi}.$$

Buffon's needle.

Sample space: possible positions of needle.
Position: center position (X, Y), orientation, Θ.



Relevant: X coordinate .. doesn't matter; Y coordinate $:=$ distance from closest line. $Y \in [0, \frac{1}{2}]$; $\Theta :=$ closest angle to vertical $[-\frac{\pi}{2}, \frac{\pi}{2}]$. When $Y \leq \frac{1}{2}\cos\Theta$: needle intersects line.

$$Pr[\text{“intersects”}] = \int_{-\pi/2}^{\pi/2} \left(Pr[\Theta \in [\theta, \theta + d\theta]] Pr[Y \leq \frac{1}{2}\cos\theta] \right)$$

$$= \int_{-\pi/2}^{\pi/2} \left([\frac{d\theta}{\pi}] \times [\frac{(1/2)\cos\theta}{1/2}] \right) = \frac{2}{\pi}[\frac{1}{2}\sin\theta]_{-\pi/2}^{\pi/2} = \frac{2}{\pi}.$$

Continuous Random Variables: CDF

$Pr[a \leq X \leq b]$ instead of $Pr[X = a]$.
For all a and b specifies the behavior!
Simpler: $P[X \leq x]$ for all x.

Cumulative probability Distribution Function of X is

$$F(x) = Pr[X \leq x]$$

$Pr[a < X \leq b] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Idea: two events $X \leq b$ and $X \leq a$.
Difference is the event $a \leq X \leq b$.

Example: CDF

Example: Bond's position.

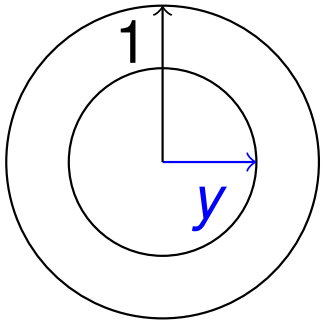
$$F(x) = Pr[X \leq x] = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{1000} & \text{for } 0 \leq x \leq 1000 \\ 1 & \text{for } x > 1000 \end{cases}$$

Probability that Bond is within 50 miles of center:

$$\begin{aligned} Pr[450 < X \leq 550] &= Pr[X \leq 550] - Pr[X \leq 450] \\ &= \frac{550}{1000} - \frac{450}{1000} \\ &= \frac{100}{1000} = \frac{1}{10} \end{aligned}$$

Example: CDF

Example: hitting random location on gas tank.
Random location on circle.



Random Variable: Y distance from center.
Probability within y of center:

$$\begin{aligned} Pr[Y \leq y] &= \frac{\text{area of small circle}}{\text{area of dartboard}} \\ &= \frac{\pi y^2}{\pi} = y^2. \end{aligned}$$

Hence,

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Calculation of event with dartboard..

Probability between .5 and .6 of center?
Recall CDF.

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

$$\begin{aligned} Pr[0.5 < Y \leq 0.6] &= Pr[Y \leq 0.6] - Pr[Y \leq 0.5] \\ &= F_Y(0.6) - F_Y(0.5) \\ &= .36 - .25 \\ &= .11 \end{aligned}$$

Density function.

Is the dart more like to be (near) .5 or .1?
Probability of "Near x" is $Pr[x < X \leq x + \delta]$.
Goes to 0 as δ goes to zero.
Try

$$\frac{Pr[x < X \leq x + \delta]}{\delta}.$$

The limit as δ goes to zero.

$$\begin{aligned}
\lim_{\delta \to 0} \frac{Pr[x < X \leq x + \delta]}{\delta} &= \lim_{\delta \to 0} \frac{Pr[X \leq x + \delta] - Pr[X \leq x]}{\delta} \\
&= \lim_{\delta \to 0} \frac{F_X(x + \delta) - F_X(x)}{\delta} \\
&= \frac{d(F(x))}{dx}.
\end{aligned}$$

Density

Definition: (Density) A **probability density function** for a random variable X with cdf $F_X(x) = Pr[X \leq x]$ is the function $f_X(x)$ where

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx.$$

Thus,

$$Pr[X \in (x, x+\delta]] = F_X(x+\delta) - F_X(x) = f_X(x)\delta.$$

Examples: Density.

Example: uniform over interval $[0, 1000]$

$$f_X(x) = F'_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{1000} & \text{for } 0 \leq x \leq 1000 \\ 0 & \text{for } x > 1000 \end{cases}$$

Example: uniform over interval $[0, \ell]$

$$f_X(x) = F'_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{\ell} & \text{for } 0 \leq x \leq \ell \\ 0 & \text{for } x > \ell \end{cases}$$

Examples: Density.

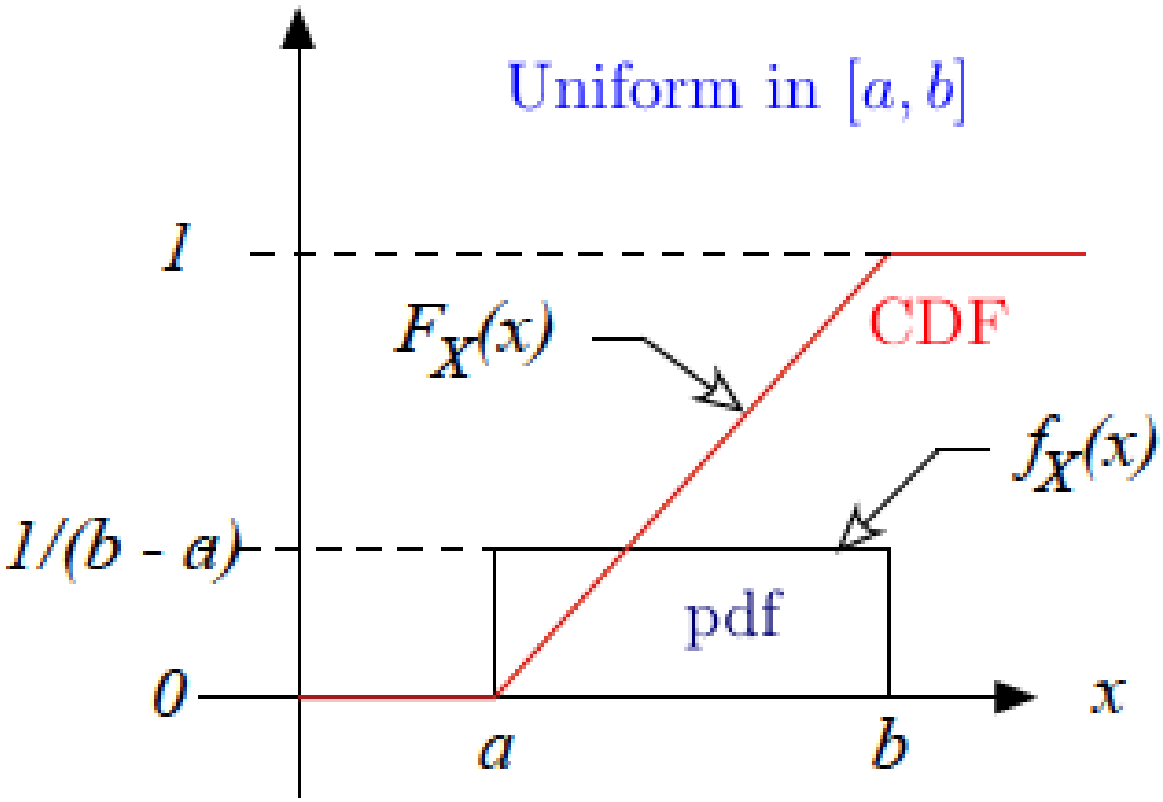
Example: “Dart” board.
Recall that

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 0 & \text{for } y < 0 \\ y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

$$f_Y(y) = F'_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{for } y > 1 \end{cases}$$

The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.
Use whichever is convenient.

$U[a, b]$

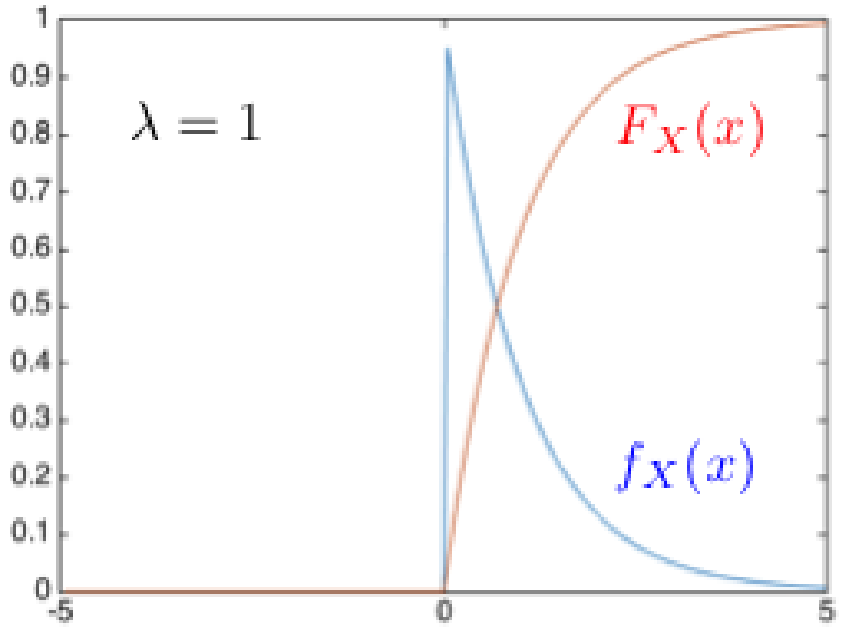


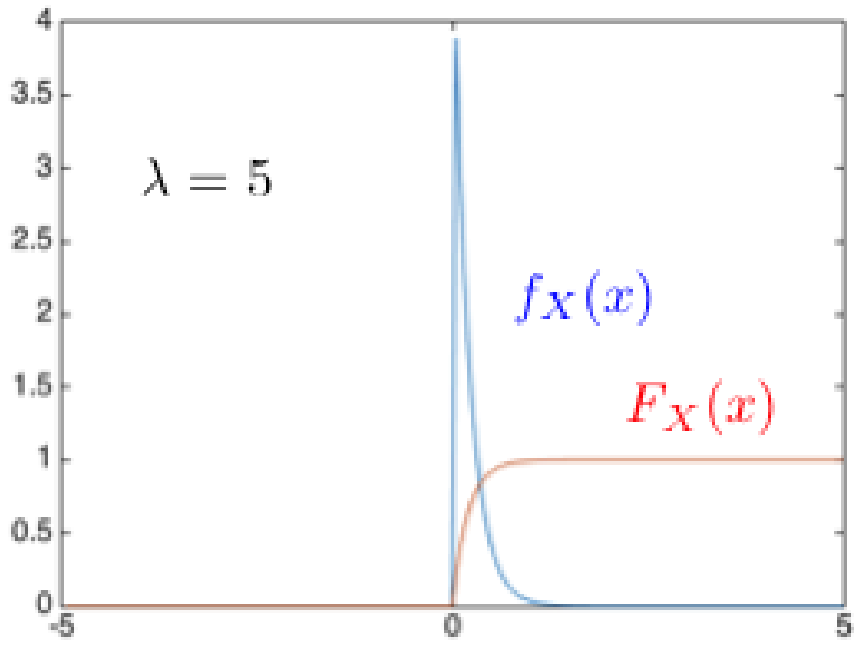
$Expo(\lambda)$

The exponential distribution with parameter $\lambda > 0$ is defined by

$$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$





Expectation

Recall that $Pr[X \in (i\delta, i(\delta+1)]] = f_X(i\delta)\delta$.
Thus,

$$\begin{aligned} E[X] &= \sum_{i=-\infty}^{\infty} (i\delta) Pr[i\delta < X \leq (i+1)\delta] \\ &= \sum_{i=-\infty}^{\infty} (i\delta) f_X(i\delta)\delta \\ &= \int_{-\infty}^{\infty} x f_X(x) dx. \end{aligned}$$

Definition The **expectation,** $E[X]$ of a continuous random variable is defined as

$$E[X] = \int_{-\infty}^{\infty} x \; f(x) dx.$$

Expectation of $U[a,b]$

Let $X = U[a,b]$. That is,

$$f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}.$$

Hence,

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \; f_X(x) dx = \int_a^b x \; \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} [\frac{x^2}{2}]_a^b \\ &= \frac{1}{2(b-a)} [b^2 - a^2] = \frac{a+b}{2}. \end{aligned}$$

Expectation: dartboard.

Example: distance from center on radius 1 dartboard.
Recall:

$$f_Y(y) = 2y1\{0 \leq y \leq 1\}.$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} y\ f(y)dy &= \int_{-\infty}^{0} 0 + \int_{0}^{1} 2y^2 dy + \int_{1}^{\infty} 0 dy \\
&= 0 + \frac{2y^3}{3}\Big|_0^1 + 0 \\
&= \frac{2}{3}
\end{aligned}$$

Try whole process for general radius. What do you get?

Expectation: Exponential.

Let $X = Expo(\lambda)$
Then,

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x\lambda e^{-\lambda x} dx \\
&= -\int_0^{\infty} x de^{-\lambda x} \\
&=^{(*)} -\{[xe^{-\lambda x}]_0^{\infty} - e^{-\lambda x} dx\} \\
&= \int_0^{\infty} e^{-\lambda x} dx = -\frac{1}{\lambda}\int_0^{\infty} de^{-\lambda x} \\
&= -\frac{1}{\lambda}[e^{-\lambda x}]_0^{\infty} = \frac{1}{\lambda}.
\end{aligned}$$

$^{(*)}$ We used the integration by parts formula:

$$\int_a^b f(x)dg(x) = [f(x)g(x)]_a^b - \int_a^b g(x)df(x),$$

which follows from $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.

Variance

Definition: The **variance** of a continuous random variable X is

$$E((X-E(X))^2) = E(X^2) - (E(X))^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} xf(x)dx\right)^2.$$

Example: uniform on $[0,\ell]$.

$$\int_0^{\ell} x^2 \frac{1}{\ell} dx = \frac{x^3}{3\ell}\Big|_0^{\ell} = \frac{\ell^2}{3}.$$

And, $E(X) = \frac{\ell}{2}$. So

$$Var(X) = \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{\ell^2}{12}.$$

$\approx \frac{n^2-1}{12}$ for uniform discrete distribution on $\{1,\ldots,n\}$.

Summary

Conditional Expectation, Continuous Probability

1. $E[Y|X] := \sum_y yPr[Y = y|X = x]$.
2. Properties: Linearity, ..., MMSE.
3. Applications: Diluting, Mixing, Going Viral, Wald.
4. Motivation for Continuous Probability: The world is continuous ....
5. pdf: $Pr[X \in (x, x+\delta]] = f_X(x)\delta$.
6. CDF: $Pr[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(y)dy$.
7. $U[a,b]$, $Expo(\lambda)$, target.
8. Expectation: $E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$.
9. Variance: $var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$.