Lecture Outline

Strengthening Induction Hypothesis.

Strong Induction

Well ordered principle.

Tutoring Option.

How does tutoring work?

1. (Ideally) You work on homework and solve (most of) it.
2. You *do not* need to write-up or turn in.
3. You read and understand homework solutions.
4. Yofu see a tutor, who gives you a short oral quiz.
   - 4.1 If you do well. Full points.
   - 4.2 Decent effort. Less than full points.
   - 4.3 Didn't understand HW solutions. Uh oh.
   - 4.4 Can try again. Limit: 2 on average.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is k^2.

Base Case 1 (0th odd number) is perfect square.

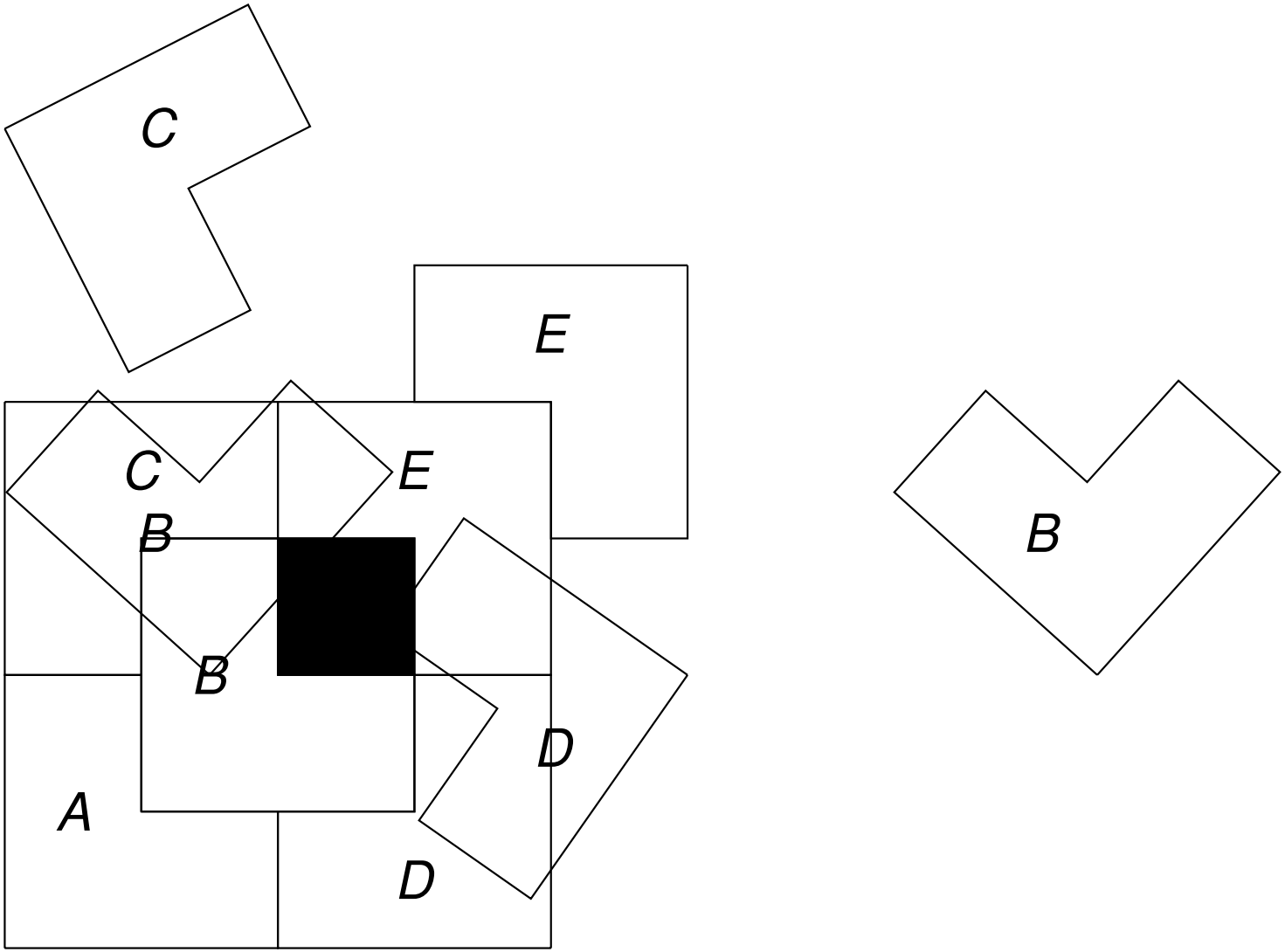
Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$
3. ??? $k^2 + 2k + 1 = (k+1)^2$
   ... P(k+1)!

$\square$

Tiling Cory Hall Courtyard.



Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a+1) \\ &= 12a + 3 + 1 \\ &= 3(4a+1) + 1 \end{aligned}$$

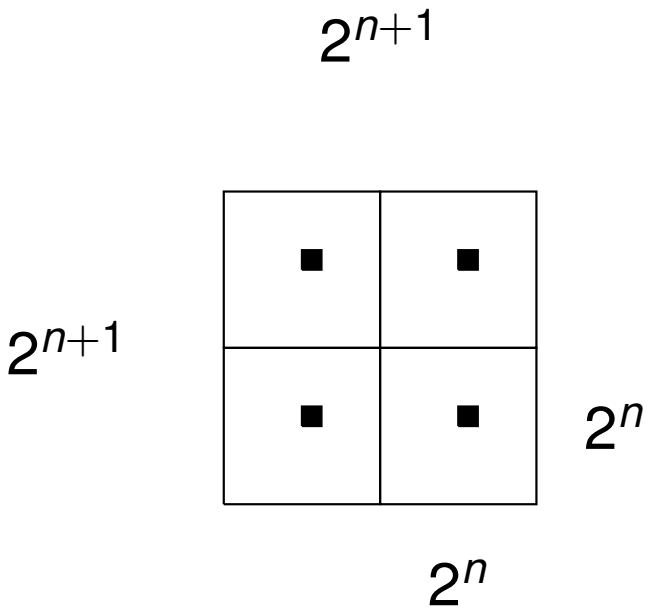
a integer $\implies$ $(4a+1)$ is an integer. $\square$

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.



What to do now???

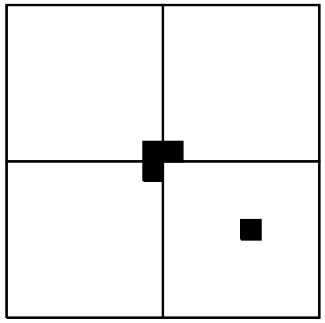
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.
Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. □

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a product of primes.

Fact: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b$ is the product of the prime factors"

i.e., the product of the factors of a and the factors of b.

Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:
"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$$
\begin{aligned}
(\forall k \in N) \quad & (Q(k) \implies Q(k+1)) \\
\equiv \quad & (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1) \\
\equiv \quad & (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))
\end{aligned}
$$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Well Ordering Principle.

If $\forall n.P(n)$ is not true, then $\exists n.\neg P(n)$.

Consider smallest m, with $\neg P(m)$,

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
I.e.,

$$\neg\forall n.P(n) \implies (\exists n, \neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle.)

But it assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: each player p plays each player q, either $p \to q$ (p beats q) or $q \to q$ (q beats q.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

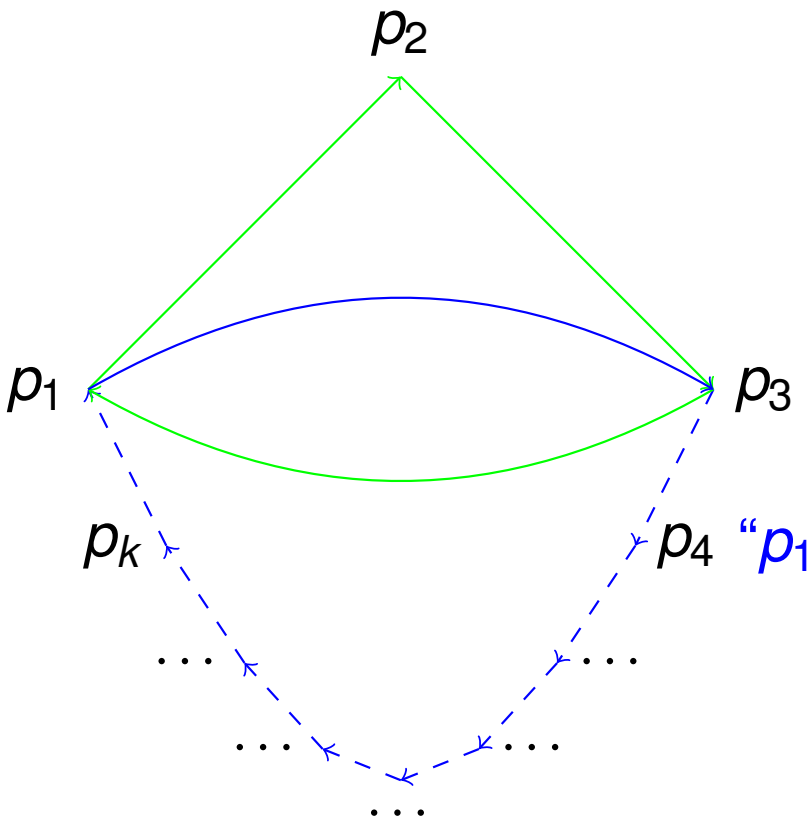
Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



"$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

"$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

Contradiction!

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

Second k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$

All k must have the same color. No horse in common! $1, 2, 3, \ldots, k, k+1$

How about $P(1) \implies P(2)$?
Fix base case.
...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Summary: principle of induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Variations:

$$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:

$$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1)))$$
$$\implies (\forall n \in N)((n \geq 1) \implies P(n))$$