Stable Marriage Problem

- Small town with n boys and n girls.
- Each girl has a ranked perference list of boys.
- Each boy has a ranked preference list of girls.

How should they be matched?

Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the couples..

- Jennifer and Brad
- Angelina and Billy-Bob

Brad prefers Angelina to Jennifer.

Angelina prefers Brad to BillyBob.

Uh..oh.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of n boy-girl pairs.

Example: A pairing $S = \{(Brad, Jen); (BillyBob, Angelina)\}$.

Definition: A **rogue couple** b, g^* for a pairing S:
b and g^* prefer each other to their partners in S

Example: Brad and Angelina are a rogue couple in S.

A stable pairing??

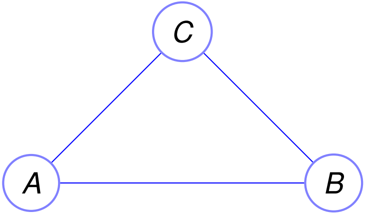
Given a set of preferences.

Is there a stable pairing?

How does one find it?

Consider a single gender version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



The Traditional Marriage Algorithm.

Each Day:

1. Each boy **proposes** to his favorite woman on his list.
2. Each girl rejects all but her favorite proposer
   (whom she puts on a string.)
3. Rejected boy crosses rejecting girl off his list.

Stop when each woman gets exactly one proposal.
Does this terminate?

...produce a pairing?

....a stable pairing?

Do boys or girls do “better”?

Example.

Boys			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Girls			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	C
2	C	B, ~~C~~	B	A,~~B~~	A
3					B

Termination.

Every non-terminated day a boy **crossed** an item off the list.

Total size of lists? n boys, n length list. n^2

Terminates in at most $n^2 + 1$ steps!

It gets better every day for girls..

Improvement Lemma:
On any day, if girl has a boy b on a string,
any future boy, b', on string is at least as good as b.

Proof:
$P(k)$- - "every day before k girl had better boy."

$P(0)$– always true as there is no day before.

Assume $P(k)$. Let b be boy **on string** on day k.

On day $k+1$, boy b comes back.

Girl can choose b just as well, or do better.

$\implies P(k+1)$. $\square$

Pairing when done.

Lemma: Every boy is matched at end.

Proof:
If not, a boy b must have been rejected n times.

Every girl has been proposed to by b,
and Improvement lemma

$\implies$ each girl has a boy on a string.

and each boy on at most one string.

n girls and n boys. Same number of each.

$\implies$ b must be on some girl's string!

Contradiction.

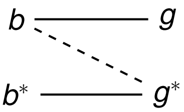
$\square$

Pairing is Stable.

Lemma: There is no rogue couple for the pairing formed by traditional marriage algorithm.

Proof:

Assume there is a rogue couple; (b, g^*)



b likes g^* more than g.

g^* likes b more than b^*.

Boy b proposes to g^* before proposing to g.

So g^* rejected b (since he moved on)

By improvement lemma, g^* likes b^* better than b.

Contradiction.

$\square$

Good for boys? girls?

Definition: A **pairing is** x**-optimal** if $x's$ partner
is its best partner in any stable pairing.

Definition: A **pairing is** x**-pessimal** if $x's$ partner
is its worst partner in any stable pairing.

Definition: A **pairing is boy optimal** if it is optimal for boys x.

..and so on for boy pessimal, girl optimal, girl pessimal.

TMA is...

Good for boys??

Theorem: TMA produces a boy-optimal pairing.

There are boys who do not get their optimal girl.

Let t be first day a boy b gets rejected by his optimal girl g from a stable pairing S.

b^* - knocks off b on day $t \implies g$ prefers b^* to b

By choice of t, b^* prefers g to optimal girl.

$\implies b^*$ prefers g to his partner g^* in S.

Rogue couple for S.

o S is not a stable pairing. Contradiction.

$\square$

Used Well-Ordering principle...again.

How about for girls?

Theorem: TMA produces girl-pessimal pairing.

T – pairing produced by TMA.

S – worse stable pairing for girl g.

In T, (g, b) is pair.

In S, (g, b^*) is pair.

g likes b^* less than she likes b.

T is boy optimal, so b likes g more than his partner in S.

Rogue couple for S

S is not stable.

Contradiction.

$\square$

Residency Matching..

The method was used to match residents to hospitals.

In dating software.

For matching jobs to servers....

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