CS70: Jean Walrand: Lecture 31.

Nonlinear Regression

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Nonlinear Regression

1. Review: joint distribution, LLSE
2. Quadratic Regression
3. Definition of Conditional expectation
4. Properties of CE
5. Applications: Diluting, Mixing, Rumors
6. CE = MMSE

Review

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Recall the non-Bayesian and Bayesian viewpoints.

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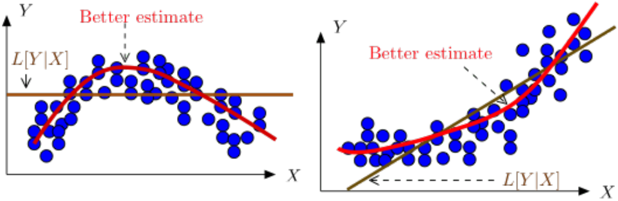

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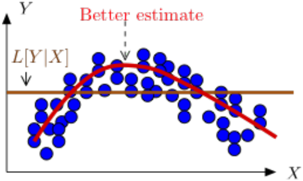

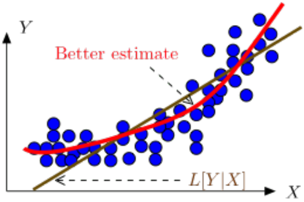

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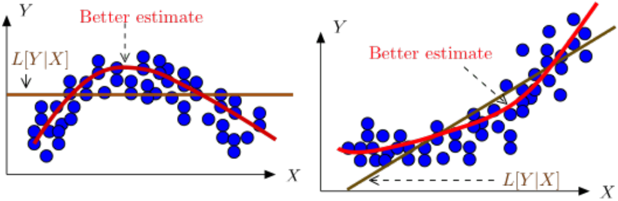

Our goal: explore estimates $\hat{Y} = g(X)$ for nonlinear functions $g(\cdot)$.

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Proof: $E[Y|X = x] = E[Y|A]$ with $A = \{\omega : X(\omega) = x\}$. □

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\text{(d)}\quad E[h(X)E[Y|X]] &= \sum_x h(x)E[Y|X=x]Pr[X=x] \\
&= \sum_x h(x)\sum_y yPr[Y=y|X=x]Pr[X=x] \\
&= \sum_x h(x)\sum_y yPr[X=x, y=y] \\
&= \sum_{x,y} h(x)yPr[X=x, y=y]
\end{aligned}$$

Properties of CE

$$E[Y|X=x]=\sum_y yPr[Y=y|X=x]$$

Theorem

(a) X,Y independent $\Rightarrow E[Y|X]=E[Y]$;
(b) $E[aY+bZ|X]=aE[Y|X]+bE[Z|X]$;
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Proof: (continued)

(e) Let $h(X) = 1$ in (d).

□

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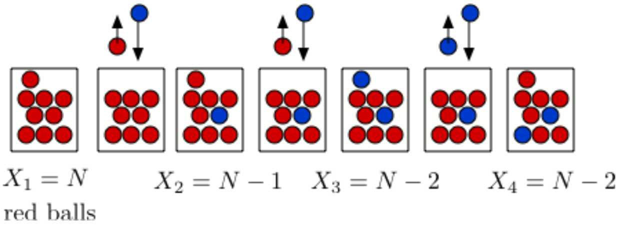
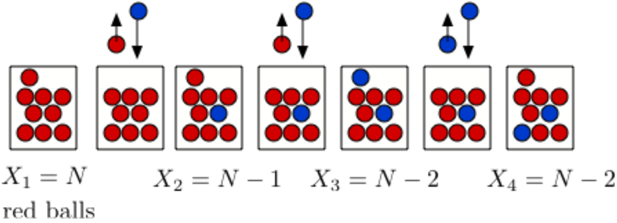
\end{aligned}$$

Application: Diluting

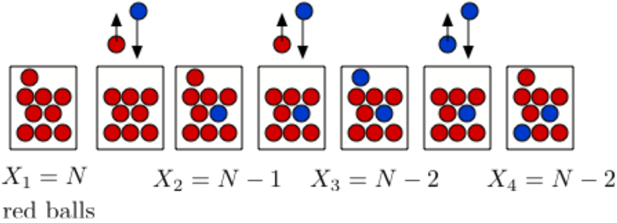


Application: Diluting



At each step, pick a ball from a well-mixed urn.

Application: Diluting



At each step, pick a ball from a well-mixed urn. Replace it with a blue ball.

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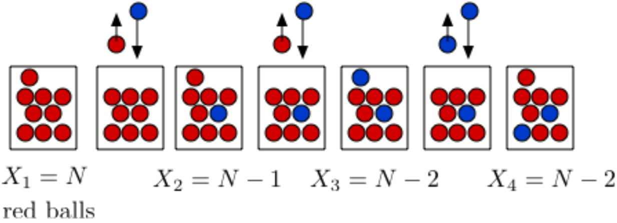
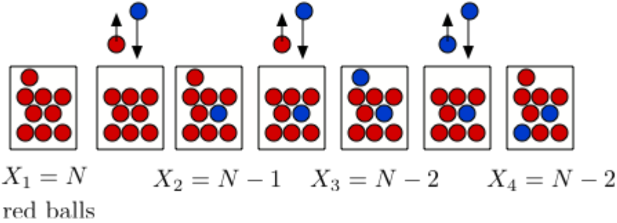


At each step, pick a ball from a well-mixed urn. Replace it with a blue ball. Let X_n be the number of red balls in the urn at step n.

Application: Diluting



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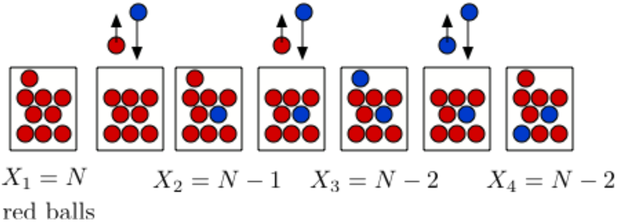

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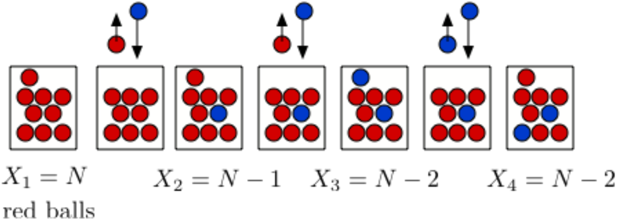
Application: Diluting



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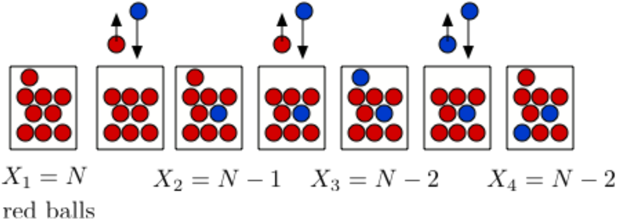
Application: Diluting



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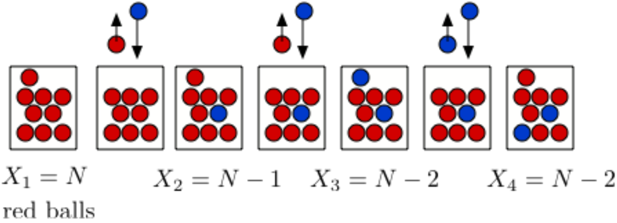

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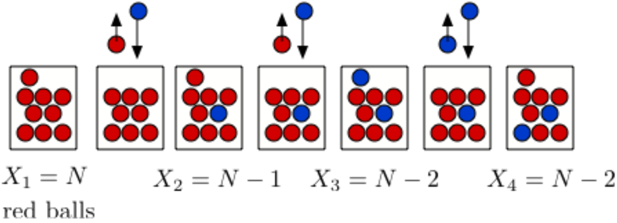

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with $\rho := (N-1)/N$.

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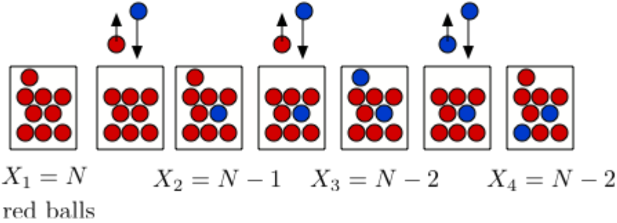

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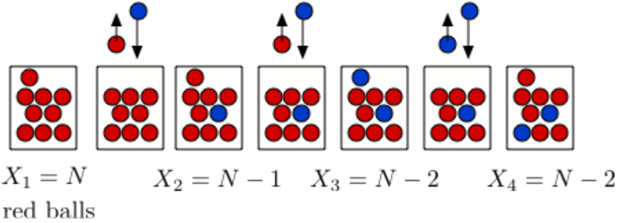

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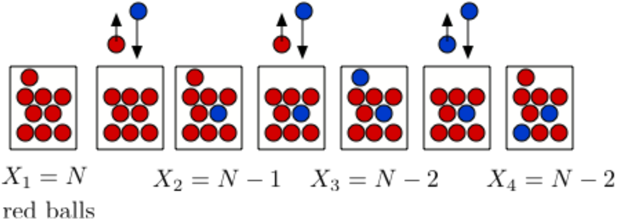

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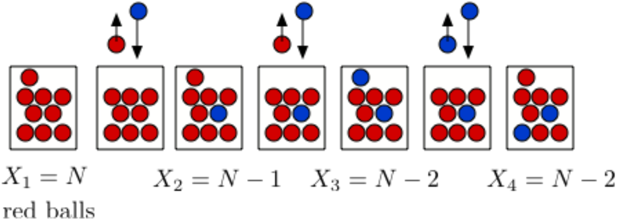

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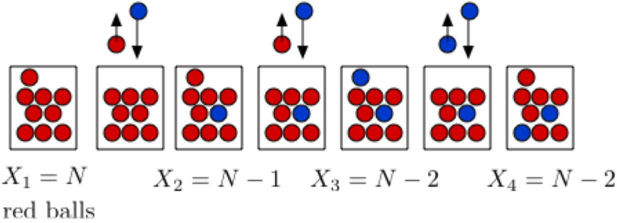

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Diluting

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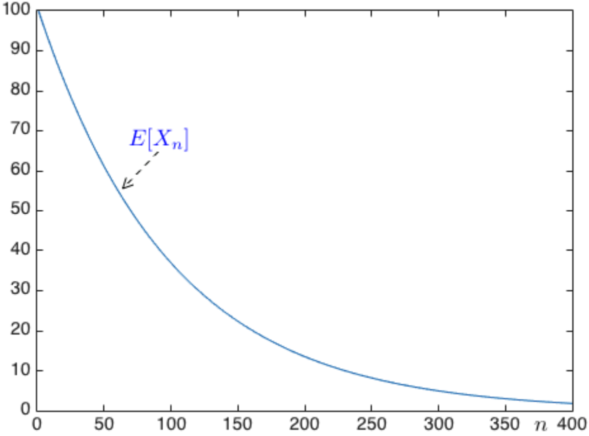

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Then, $X_n = Y_n(1) + \cdots + Y_n(N)$.

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$$\begin{aligned} E[X_n] &= E[Y_n(1) + \cdots + Y_n(N)] = NE[Y_n(1)] \\ &= NPr[Y_n(1) = 1] \end{aligned}$$

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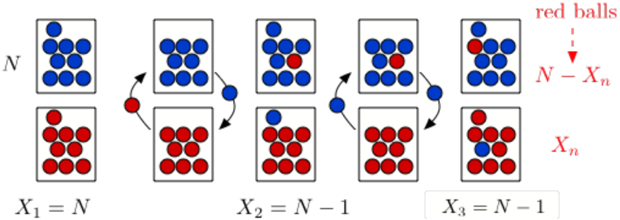
Application: Mixing



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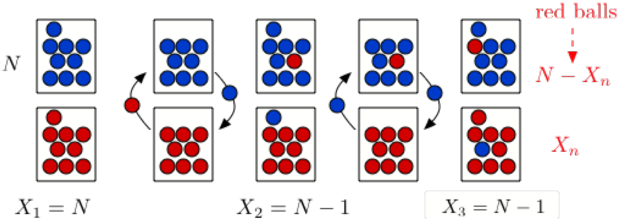
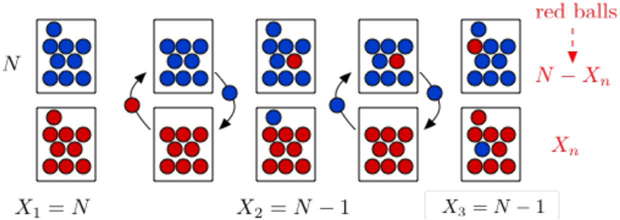


At each step, pick a ball from each well-mixed urn.

Application: Mixing



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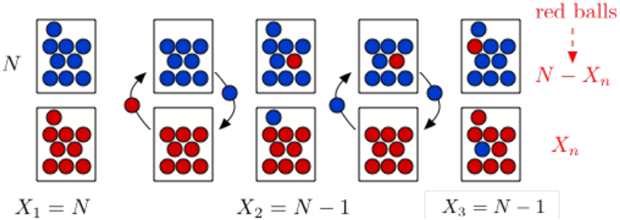

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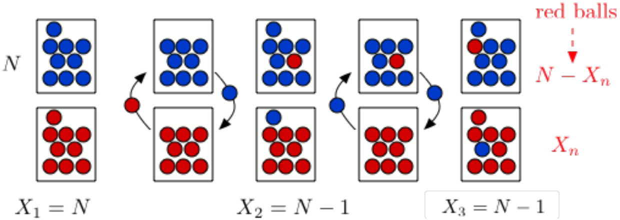

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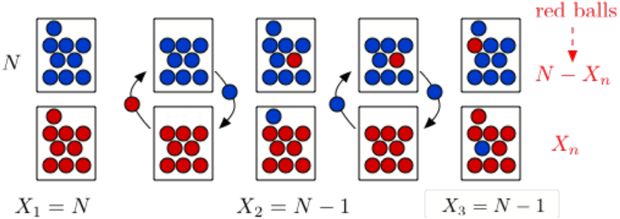

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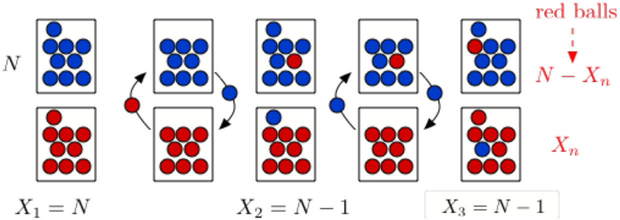
Application: Mixing



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where $p = (1 - m/N)^2$ (B goes up, R down)

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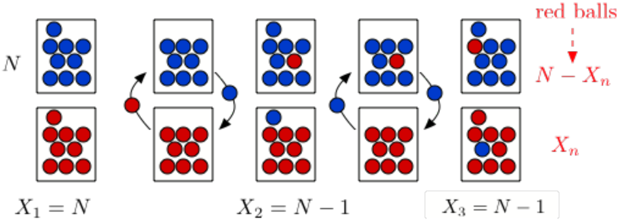

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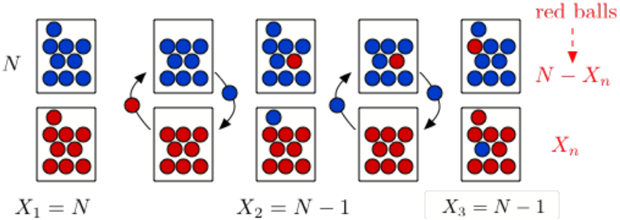

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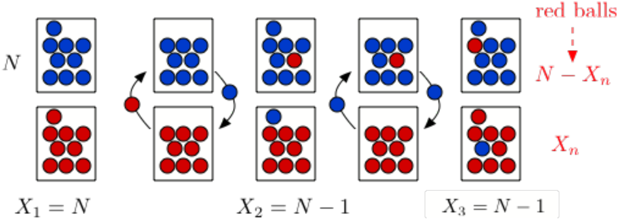
Thus,
$E[X_{n+1}|X_n] = X_n + p - q$

Application: Mixing



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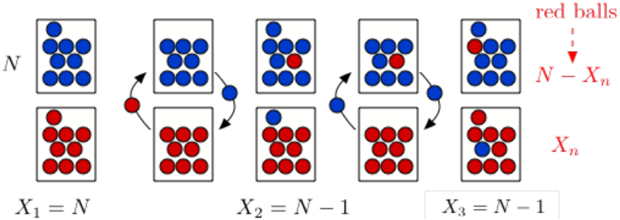

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Thus,
$E[X_{n+1}|X_n] = X_n + p - q = X_n + 1 - 2X_n/N = 1 + \rho X_n, \ \rho := (1 - 2/N)$.

Mixing

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\end{aligned}$$

Hence,

$$E[X_n] = \frac{1 - \rho^{n-1}}{1 - \rho} + \rho^{n-1} N, n \geq 1.$$

Application: Mixing

Here is the plot.

Application: Mixing

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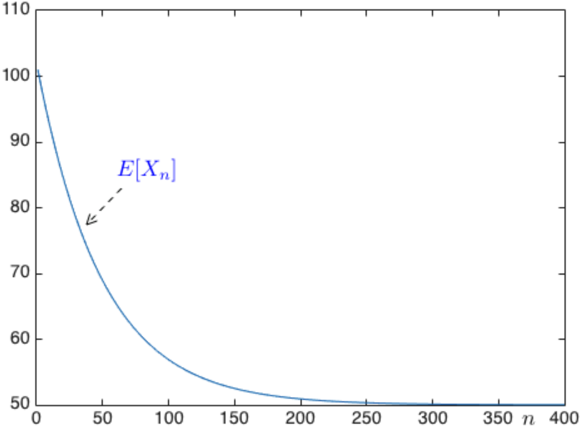

Application: Going Viral

Consider a social network (e.g., Twitter).

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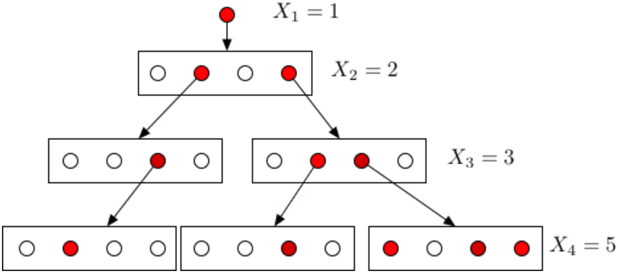

Application: Going Viral

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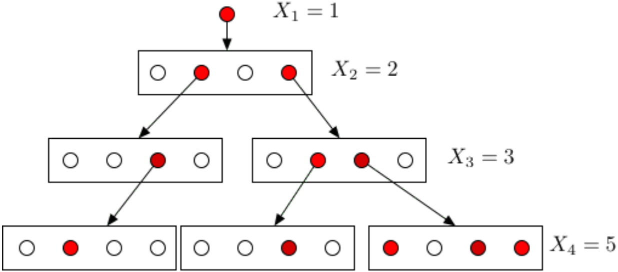
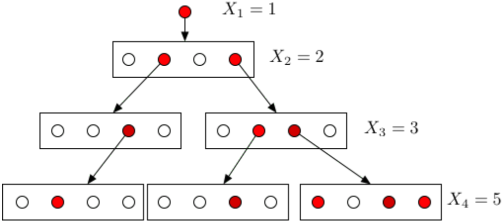
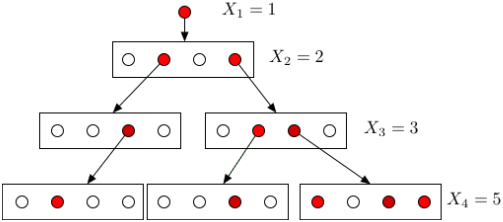


In this example, $d = 4$.

Application: Going Viral

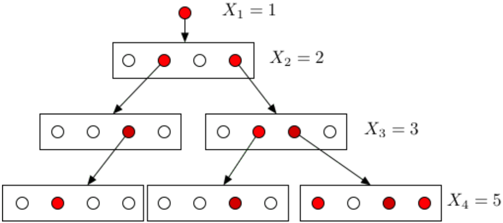


Application: Going Viral



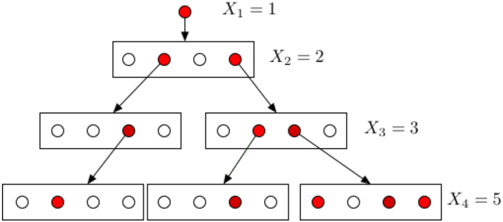
Fact:

Application: Going Viral



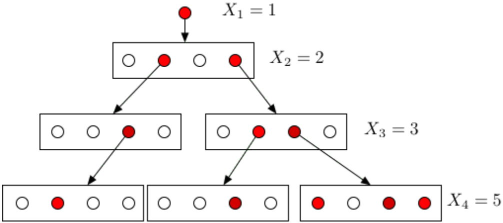
Fact: Let $X = \sum_{n=1}^{\infty} X_n$.

Application: Going Viral



Fact: Let $X = \sum_{n=1}^{\infty} X_n$. Then, $E[X] < \infty$ iff $pd < 1$.

Application: Going Viral



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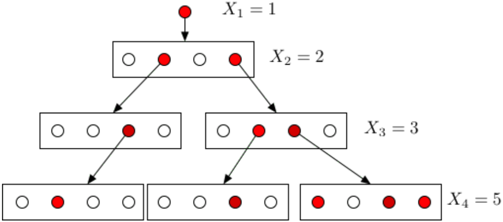
Given $X_n = k$, $X_{n+1} = B(kd, p)$.

Application: Going Viral



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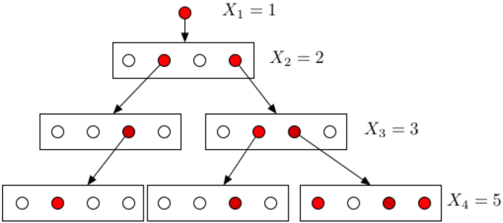
Given $X_n = k$, $X_{n+1} = B(kd, p)$. Hence, $E[X_{n+1} | X_n = k] = kpd$.

Application: Going Viral



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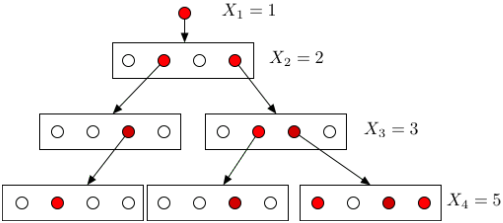
Application: Going Viral



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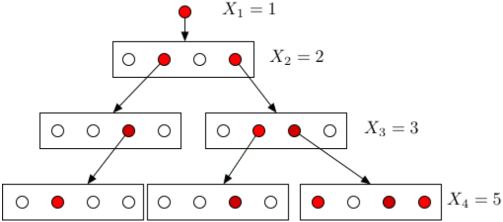
Application: Going Viral



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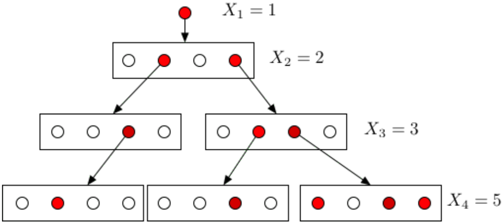
If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1}$

Application: Going Viral



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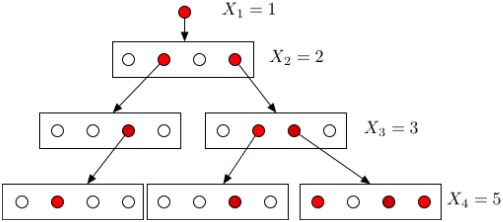

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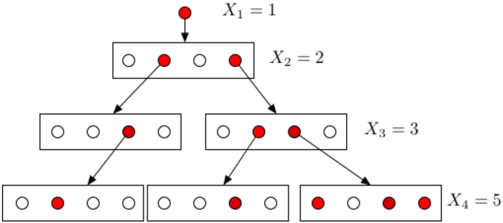
If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1} \implies E[X] \leq (1 - pd)^{-1}$.

If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C. \qquad \square$$

Application: Going Viral



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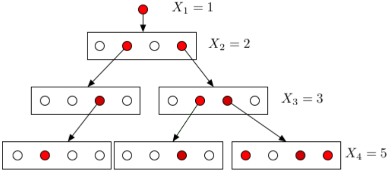
In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

Application: Going Viral



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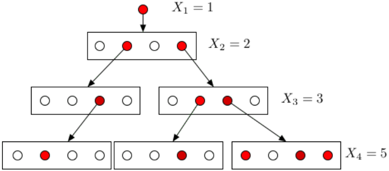
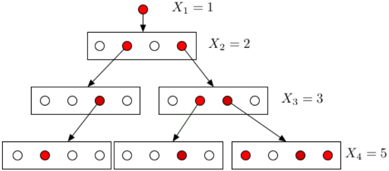


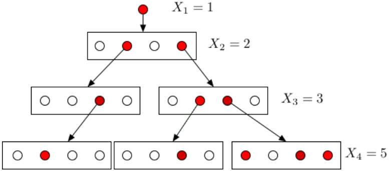
An easy extension:

Application: Going Viral



An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$.

Application: Going Viral



An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$. Then, the same fact holds.

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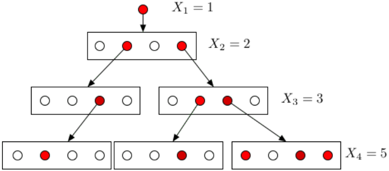

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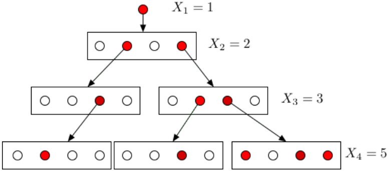

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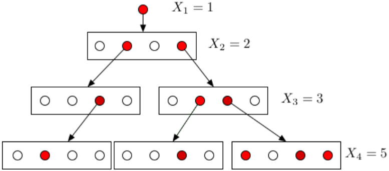

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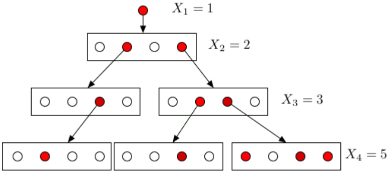

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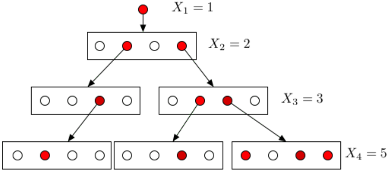
Application: Going Viral



An easy extension: Assume that everyone has an independent number D_i of friends with $E[D_i] = d$. Then, the same fact holds.

To see this, note that given $X_n = k$, and given the numbers of friends $D_1 = d_1, \ldots, D_k = d_k$ of these X_n people, one has $X_{n+1} = B(d_1 + \cdots + d_k, p)$. Hence,

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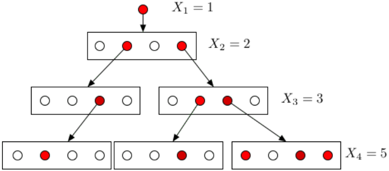

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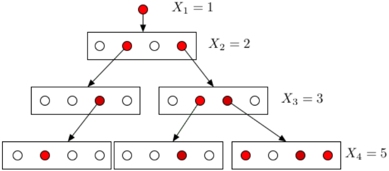

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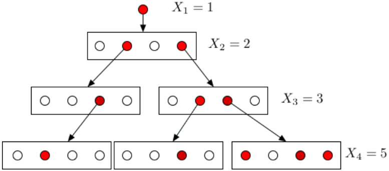

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We conclude as before.

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Hence, $E[X_1 + \cdots + X_Z] = E[\mu Z] = \mu E[Z].$ □

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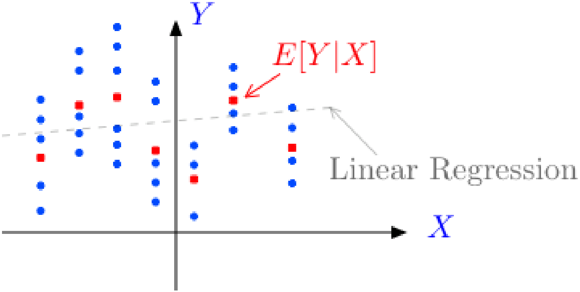

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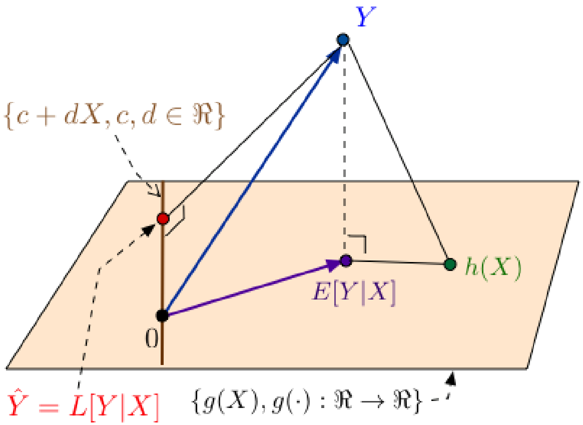
Thus, $E[(Y-h(X))^2] \geq E[(Y-g(X))^2]$. □

$E[Y|X]$ and $L[Y|X]$ as projections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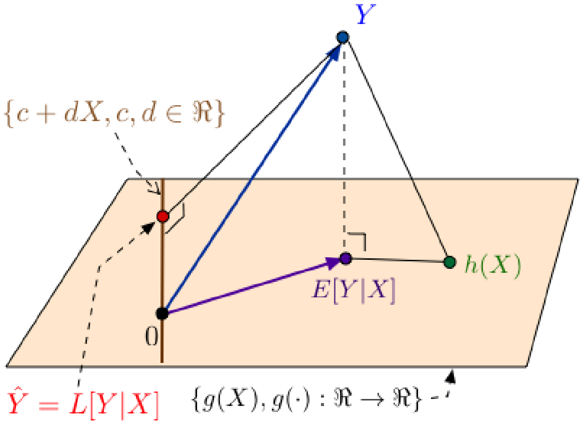
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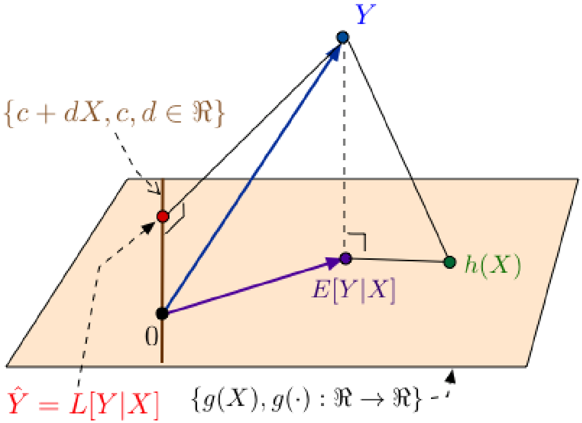

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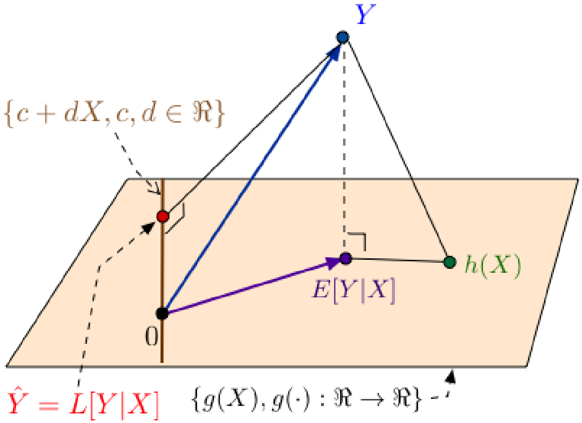
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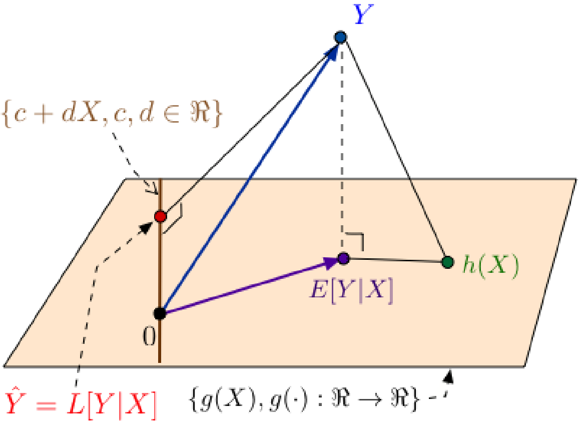
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