CS 70 Discrete Mathematics for CS Spring 2006 Vazirani — Homework 1

1. **(5 pts.) Exclusive OR** The exclusive OR (written as XOR or $\oplus$) is just what it sounds like: $P \oplus Q$ is true when exactly one of P, Q is true.

   Show, using a truth table, that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \neg(P \wedge Q)$.

2. **(15 pts.) Propositions**
   For each of the following propositions: (a) Explain in English what the statement says. (b) Negate the proposition (c) Say whether the proposition is true or false. Justify your answer.

   (a) $(\forall x \in \mathbf{Z})((x < \sqrt{48}) \vee (x > 7))$

   (b) $(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(\exists \mathbf{z} \in \mathbf{R})(x < z < y)$

   (c) $(\forall x \in \mathbf{Z})(\exists y \in \mathbf{Z})(x - y > 16)$

   (d) $(\exists x \in \mathbf{R})(\forall y \in \mathbf{R})(xy = 0)$

   (e) $(\forall n \in \mathbf{N})((n \text{ not prime} \wedge n > 1) \implies (\exists p \in \mathbf{N})(p \text{ prime} \wedge p|n))$

3. **(10 pts.) Portia's Caskets**
   In Shakespeare's Merchant of Venice, Portia had three caskets, gold, silver and lead, inside one of which was her portrait. Her suitor was asked to choose one of the caskets, and if he chose the portrait, he could claim Portia as his bride.

   Here is Portia's casket test again with a twist. Suppose there are two casket makers, Bellini and Cellini. Bellini always writes true statements on his caskets, while Cellini always writes false ones.

   (a) Suppose one of the caskets contains, not a portrait, but a dagger, and the suitor's job is to avoid choosing that casket. The caskets are inscribed as follows:

   Gold: The dagger is in this casket

   Silver: This casket is empty.

   Lead: At most one of these three caskets was made by Bellini.

   Which casket should the suitor choose? Explain your answer.

   (b) Suppose Portia placed her portrait in one of the caskets and the suitor's job is to select the casket with the portrait and to determine the maker of each of the three caskets. The caskets are inscribed as follows:

   Gold: The portrait is in here.

   Silver: The portrait is in here.

   Lead: At least two of these caskets were made by Cellini.

   How should the suitor answer to pass the test? Explain.

4. **(20 pts.) Prove the following statements:**

   (a) If a a positive integer then a^2 is even if and only if a is even.

(b) Later in the course, we will prove a more general statement: If a, b are positive integers and p is a prime then if $p|ab$, $p|a$ or $p|b$. Show that this statement is false if p is not a prime. (Note $a|b$ means "a is a divisor of b.")

(c) For a positive integer n, $\sqrt{n}$ is rational if and only if n is a perfect square. (A positive integer m is a perfect square if there exists an integer j such that $m = j^2$.)

5. (10 pts.) Game from Lecture Recall the two-player game from class: The first player picks a number n between 1 and 100 and then chooses a divisor d of n and subtracts it from n, passing the result to the other player. That player, too, must choose a divisor and subtract it, returning the result to player 1. The players are allowed to choose 1 as a divisor, but not the number itself. The game continues until one of the players receives 1 and can no longer move. That player loses.

(a) State a non-trivial theorem about this game.

(b) Argue informally that your theorem is correct making sure to explain the three key points noted in lecture. (You will argue it more formally on next week's homework, after we learn about induction.)