CS 70 Discrete Mathematics for CS Spring 2006 Vazirani — HW 2

1. **(20 pts.)** The following problem is taken from Lewis Carroll's Symbolic Logic of 1896: i) No one, who is going to a party, ever fails to brush his or her hair. ii) No one looks fascinating, if he or she is untidy. iii) Opium-eaters have no self-command. iv) Everyone who has brushed his or her hair looks fascinating. v) No one wears kid gloves, unless he or she is going to a party. vi) A person is always untidy if he or she has no self-command.

   Notation: $U = \{Persons\}$

   $P(x) =$ "x is going to a party."

   $H(x) =$ "x has brushed his or her hair."

   $S(x) =$ "x has self-command."

   $F(x) =$ "x looks fascinating."

   $O(x) =$ "x is an opium eater."

   $T(x) =$ "x is tidy."

   $G(x) =$ "x is wearing white kid gloves."

   (a) Write each of the sentences i - vi in quantified form, using the universe and open sentences defined above.

   (b) Rewrite each of the 6 quantified sentences, this time using the contrapositive of the if-then clause.

   (c) Given the above 12 quantified sentences, what can you conclude about someone that wears kid gloves? Write the sentences in order that lead you to your conclusion.

2. **(30 pts.) Simple induction**

   (a) Prove that for each $n \in \mathbf{N}$, $3 + 11 + 19 + \cdots + (8n-5) = 4n^2 - n$.

   (b) Prove that $3^n < n!$ for all integers $n > 6$.

   (c) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for all integers $n > 0$.

3. **(20 pts.) Making big rocks into little rocks**
   In the mound-splitting game, one starts off with a single mound of pebbles. In each move, one picks a mound, splits it into two mounds of arbitrary size (say, k and m pebbles), multiplies the number of pebbles in each of the two new mounds, and writes down the result (i.e., $k \times m$). One keeps playing until there is no mound with more than one pebble, and then one adds up all of the numbers previously written down. Prove that if one starts with a mound of n pebbles, the final sum will be $n(n-1)/2$, *no matter what order the moves are done in*.

4. (20 pts.) Tower of Brahma

This game is also known as the Towers of Hanoi or the End of the World Puzzle. It was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

Use and induction proof to estabish the number of moves required in general if there are n disks on the original needle. Assuming that the priests can move a disk each second, roughly how long does the prophecy predict before the destruction of the World?

5. (20 pts.) Proofs to Grade

Assign a grade of A (correct), C (partially correct) or F (failure) to the following proof. If you give a grade other than A, please explain exactly what is wrong with the structure or the reasoning in the *proof*. You should justify all your answers (Remember, saying that the claim is false is *not* a justification).

(a) **Claim:** For every $n \in \mathbf{N}$, n^2+n is odd. Base Case: The natural number 1 is odd. Inductive Step: Suppose $k \in \mathbf{N}$ and k^2+k is odd. Then,

$$(k+1)^2+(k+1) = k^2+2k+1+k+1 = (k^2+k)+(2k+2)$$

is the sum of an odd and an even integer. Therefore, $(k+1)^2+(k+1)$ is odd. By the Principle of Mathematical Induction, the property that n^2+n is odd is true for all natural numbers n. ♠

(b) **Claim:** $(\forall n \in \mathbf{N})(7^n = 1)$.

Proof: (uses strong induction)

Base Case: Certainly $7^0 = 1$. Inductive step: Now suppose that $7^j = 1$ for all $0 \leq j \leq k$.

We need to show that $7^j = 1$ for all $j, 0 \leq j \leq k+1$. To do this, we need only to prove that $7^{k+1} = 1$. But,

$$7^{k+1} = \frac{(7^k \cdot 7^k)}{7^{k-1}} = \frac{(1 \cdot 1)}{1} = 1$$

Hence, by the Principle of Strong Induction, $7^m = 1$ for all $m \in \mathbf{N}$. ♡