CS 70 Discrete Mathematics for CS
Spring 2006 Vazirani

HW 11

Due Friday April 28

1. **Random bit strings**
   Consider a random bit string S of length n.

   (a) For a given position j in S, what is the probability that it is a starting point of a run of at least k ones?

   (b) What is the expected number of places j at which runs of at least k ones start?

   (c) Use Markov's inequality to show that the probability that there exists a run of at least $c \log n$ ones is less than $\frac{1}{n^{c-1}}$.

   (d) We now consider runs of alternating ones and zeroes that start with a one (e.g. 101010). What is the expected number of places j at which alternating runs that begin with a one and have at least k bits start?

2. **How to beat the heat**
   It's a hot summer day in the Central Valley. Three children Alice, Bob, and Carlos are engaged in a three-way duel with water balloons. They start by drawing lots to determine who throws first, second, and third, then take their places at the corners of an equilateral triangle. They agree to throw single water balloons in turn and continue in the same cyclic order until two of them have been soaked. Each player may throw at any other in his or her turn. You should assume the following: all the children have an essentially infinite supply of ammunition; a water balloon explodes on contact, drenching its target (who then leaves the game); when a water balloon misses its target, it explodes far enough away not to get anyone wet. Figuring out the best strategy for each player and their chance of winning can be quite hard in general. In this question we will analyze a particular instance of this problem.

   Suppose all three know that Alice always hits her target, Bob is 75% accurate, and Carlos is 50% accurate. Of course, if for some reason any of them deliberately decides to miss they can do so with certainty.

   (a) Suppose the duel involves only Alice and Bob (say Carlos has already been eliminated), with Bob taking the first shot. What is the chance that Bob wins?

   (b) Suppose the duel involves only Bob and Carlos, with Carlos taking the first shot. What is the chance that Carlos wins?

   (c) Suppose the duel involves only Alice and Carlos, with Carlos taking the first shot. What is the chance that Carlos wins?

   (d) Now going back to the original duel, suppose that Carlos has drawn the first shot, and Bob second. What is Carlos' best strategy, and what is the chance that he comes out the eventual winner? What is Alice's chance of coming out the eventual winner?

3. Expectation

(a) What is the average number of dots shown by a six-sided die thrown once at random?

(b) You wish to maximize the value shown by the die. If you do not like the outcome of your first throw you are now allowed to discard it and instead rethrow the die a second time. When should you do so?

(c) What is the average number of dots on the throw that you keep above? i.e. what is the average number of dots you achieve using the strategy you chose above.