CS 70 Discrete Mathematics for CS
Spring 2006 Vazirani

Practice Problems

1. **(10 pts.) Modular Arithmetic**
   Let $a_1, a_2, \ldots, a_{10}$ be positive integers such that none of them is divisible by 11.

   Show that $x = (a_1)^{10} + (a_2)^{10} + \ldots + (a_{10})^{10}$ is not divisible by 11.

2. **(15 pts.) Induction**
   Prove that if $x + 1/x$ is an integer, then $\forall n \geq 1,\ \ x^n + 1/x^n$ is an integer.

3. **(10 pts.) Cardinality**
   Recall that a real number which is not a rational number is called irrational. Are the irrationals countably infinite or uncountably infinite? Prove your answer. (You may use the fact that the reals are uncountable).

4. **(10 pts.) More Cardinality**
   Prove that the number of computer programs in Java is countable.

5. **(10 pts.) Expectation**
   Suppose we pick a subset of the set $\{1, 2, \ldots, n\}$ uniformly at random (i.e. each of the 2^n subsets is equally likely).

   What is the expected sum of all the elements in the subset?

   (e.g. if the subset is $\{2, 5, 9, 15\}$, then the sum is 31.)

6. **(10 pts.) Conditional Probability**
   Each day you decide with probability $1/4$ to wait for the bus and probability $3/4$ to walk to school. If you wait for the bus the chances are $3/4$ that you get to class on time. Whereas if you walk the chance is only $1/2$.

   - What is the chance that you make it to class on time?
   - Given that you made it to class on time today, what is the chance that you took the bus?

7. **(15 pts.) Probability**

   (a) Suppose we successively pick two numbers uniformly and independently from the set $\{0, 1, \ldots, n\}$. What is the probability that the sum of the two numbers is $\leq n$?

   (b) Now suppose we successively pick three numbers uniformly and independently from the set $\{0, 1, \ldots, n\}$. What is the probability that the sum of the three numbers is *exactly* n?

8. **(20 pts.) Random Variables**
   A random variable X has expectation $\mathbf{E}[X] = 2$ and variance $Var[X] = 9$, and the value of X never exceeds 10.

   Circle those **two** of the following statements that *must* be true about X. For each statement give a one sentence explanation for why the statement must be true or why it could be false.

- $P[X = 2] > 0$
- $P[X \geq 2] = P[X \leq 2]$
- $E[X^2] = 13$
- $P[X \geq 6] \leq 1/3$
- $P[X \geq 6] \leq 9/16$

9. (pts.) Geometric Distribution

The starship Almost Invinsible is cruising across the galaxy. Each day there is a 1 in 80 chance that the ship is blown up by Arcturian raiders, a 1 in 300 chance that it is eaten by a giant spaceworm and 1 in 2000 chance that it is sucked into a black hole. Assume that these events are independent. What is the expected number of days of the Almost Invincible's cruise?

10. (pts.) Normal Distribution

A political scientist wishes to determine if there is a significant difference between the preferences of voters in two similar neighborhoods of a city with respect to an upcoming race for mayor. Samples of 30 voters are taken from each of the neighborhoods. In one sample 12 voters prefer the incumbent while in the other neighborhood 19 do so. Using a 10% significance level (1.65 σ), decide whether there is a significant difference between the neighborhoods.

11. (pts.) Poisson Distribution

In a racially integrated school, 15% of the students are non-white. What is the probability that a randomly chosen class of 20 students is all-white?