Problem Set #7

1. **Count the Square-frees**
   A positive integer is called *square-free* if it is not divisible by the square of any positive integer greater than 1. For example $35 = 5 \cdot 7$ is square-free but $18 = 2 \cdot 3^2$ is not. 1 is square-free.
   Use *inclusion-exclusion* to find the number of square-free positive integers strictly less than 201.

2. **Count Special Numbers**
   Consider strings of digits of length n: i.e. the set $\{0, 1, 2, \ldots, 9\}^n$.

   (a) How many strings are there where the digits (as numbers) add up to an even number?

   (b) A *palindrome* is a string that is equivalent when read backwards. Hence 12321 is a palindrome, while 11991 is not. Assuming that n is even, how many palindromes are there of length n?

   (c) Somewhat combining the previous two questions: How many digit strings are there of odd length n that are palindromes and of which the digits add up to an even number?

3. **King Positioning**
   The king positioning in an arrangement of a deck of 52 playing cards is the sequence of numerical positions in the deck, from one to 52, of the four successive kings. For example, the king positioning $(1, 2, 3, 4)$ means all the kings come at the beginning of the deck. The king positioning $(1, 18, 35, 52)$ describes the situation in which the kings are spaced uniformly-with exactly 16 cards between successive kings.

   (a) How many king positionings are there?

   (b) How many king positionings are there in which no two kings are adjacent?

   (c) Of the 52! possible arrangements of the deck, how many have no two kings adjacent?

4. **Combinatorial vs. Algebraic Proofs**
   Prove the following identity both by

   (a) Algebraic manipulation.

   (b) A combinatorial argument.

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

5. **Monty Hall's Game**
   We learned about the Monty Hall's game in the lecture. Now consider a variation of Monty Hall's game: The contestant still picks one of three doors, with a prize randomly placed behind one door and goats behind the other two. But now, instead of always opening a door to reveal a goat, Monty instructs Carol to always open a door when the contestant is right and half the time when he's wrong, then, conditioned of being in a game where Monty offers a switch, what is the probability of winning by staying and what is the probability of winning by switching? In other words, the contestant is *not* always given the option of switching when he's wrong.