Problem Set 8

1. **Independence**
   Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two *independent* events. Let $\overline{A} = \Omega - A$ and $\overline{B} = \Omega - B$ (sometimes written $\neg A$ and $\neg B$) denote the complementary events.

   For the purposes of this question, you may use the following definition of independence: Two events A, B are *independent* if $\Pr[A \cap B] = \Pr[A]\Pr[B]$.

   (a) (2 pts) Prove or disprove: $\overline{A}$ and $\overline{B}$ are necessarily independent.

   (b) (2 pts) Prove or disprove: A and $\overline{A}$ are necessarily independent.

   (c) (2 pts) Prove or disprove: It is possible that $A = B$.

2. **Burnt pancakes** (2 pts)
   I have a bag containing three pancakes: One golden on both sides, one burnt on both sides, and one golden on one side and burnt on the other. You shake the bag, draw a pancake at random, look at one side, and notice that it is burnt. What is the probability that the other side is burnt? Show your work.

3. **Poisoned pancakes**
   You've been hired as an actuary by IHOP corporate headquarters, and have been handed a report from corporate intelligence that indicates that a covert team of ninjas hired by Denny's will sneak into some IHOP, and will have time to poison 5 of the pancakes being prepared (they can't stay any longer to avoid being discovered by Pancake Security). Given that an IHOP kitchen has 50 pancakes being prepared, and there are 10 patrons, each ordering 5 pancakes (which are chosen uniformly at random from the pancakes in the kitchen), calculate:

   (a) (4 pts) The probability that a particular patron:

      i. will not receive any poisoned pancakes

      ii. will receive exactly 1 poisoned pancake

      iii. will receive at least one poisoned pancake

   (b) (2 pts) The expected number of patrons that will survive the morning (assuming that 2 poisoned pancakes are needed to kill a person).

4. **Why do we consider "every subset" for mutual independence?** (3 pts)
   In lecture we showed that, if $A_1, \ldots, A_n$ are mutually independent, then $\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1] \times \Pr[A_2] \times \ldots \times \Pr[A_n]$. Give a specific counter-example (over a probability space of your choice) to show that the converse of this theorem does not hold.

5. **Royal expectations** (3 pts)
   Let's take another look at the adjacent-kings problem from HW7 from the point of view of expectations.

   We say that a deck has k *king adjacencies* if k of the kings are followed by another king. That is, if, in a particular deck arrangement, all the kings are next to each other, there are 3 king adjacencies.

   What is the expected number of king adjacencies in a standard 52-card deck, shuffled into a permutation uniformly at random? Justify your answer.