CS 70	Discrete Mathematics for CS	
Spring 2007	Luca Trevisan	Lecture 14

Hypercubes

Recall that the set of all n-bit strings is denoted by $\{0,1\}^n$. The n-dimensional hypercube is a graph whose vertex set is $\{0,1\}^n$ (i.e. there are exactly 2^n vertices, each labeled with a distinct n-bit string), and with an edge between vertices x and y iff x and y differ in exactly one bit position. i.e. if $x = x_1 x_2 \dots x_n$ and $y = y_1 y_2 \dots y_n$, then there is an edge between x and y iff there is an i such that $\forall j \neq i, x_j = y_j$ and $x_i \neq y_i$.

There is another equivalent recursive definition of the hypercube:

The n-dimensional hypercube consists of two copies of the $n-1$-dimensional hypercube (the 0-subcube and the 1-subcube), and with edges between corresponding vertices in the two subcubes. i.e. there is an edge between vertex x in the 0-subcube (also denoted as vertex $0x$) and vertex x in the 1-subcube.

Claim: The total number of edges in an n-dimensional hypercube is $n2^{n-1}$.

Proof: Each vertex has n edges incident to it, since there are exactly n bit positions that can be toggled to get an edge. Since each edge is counted twice, once from each endpoint, this yields a grand total of $n2^n/2$.

Alternative Proof: By the second definition, it follows that $E(n) = 2E(n-1) + 2^{n-1}$, and $E(1) = 1$. A straightforward induction shows that $E(n) = n2^{n-1}$.

We will prove that the n-dimensional hypercube is a very robust graph in the following sense: consider how many edges must be cut to separate a subset S of vertices from the remaining vertices $V-S$. Assume that S is the smaller piece; i.e. $|S| \leq |V-S|$.

Theorem: $|E_{S,V-S}| \geq |S|$.

Proof: By induction on n. Base case $n = 1$ is trivial.

For the induction step, let S_0 be the vertices from the 0-subcube in S, and S_1 be the vertices in S from the 1-subcube.

Case 1: If $|S_0| \leq 2^{n-1}/2$ and $|S_1| \leq 2^{n-1}/2$ then applying the induction hypothesis to each of the subcubes shows that the number of edges between S and $V-S$ even without taking into consideration edges that cross between the 0-subcube and the 1-subcube, already exceed $|S_0| + |S_1| = |S|$.

Case 2: Suppose $|S_0| > 2^{n-1}/2$. Then $|S_1| \leq 2^{n-1}/2$. But now $|E_{S,V-S}| \geq 2^n - 1 \geq |S|$. This is because by the induction hypothesis, the number of edges in $E_{S,V-S}$ within the 0-subcube is at least $2^{n-1} - |S_0|$, and those within the 1-subcube is at least $|S_1|$. But now there must be at least $|S_0| - |S_1$ edges in $E_{S,V-S}$ that cross between the two subcubes (since there are edges betwen every pair of corresponding vertices. This is a grand total of $2^{n-1} - |S_0| + |S_1| + |S_0| - |S_1| = 2^{n-1}$.

Hamiltonian Tours and Paths

A Hamiltonian tour in an undirected graph $G = (V,E)$ is a cycle that *goes through every vertex exactly once*. A Hamiltonian path is a path that goes through every vertex exactly once.

Theorem: For every $n \geq 2$, the n-dimensional hypercube has a Hamiltonian tour.

Proof: By induction on n. In the base case $n = 2$, the 2-dimensional hypercube, the length four cycle starts from 00, goes through 01, 11, and 10, and returns to 00.

Suppose now that every $(n-1)$-dimensional hypercube has an Hamiltonian cycle. Let $v \in \{0,1\}^{n-1}$ be a vertex adjacent to 0^{n-1} (the notation 0^{n-1} means a sequence of $n-1$ zeroes) in the Hamiltonian cycle in a $(n-1)$-dimensional hypercube. The following is a Hamiltonian cycle in an n-dimensional hypercube: have a path that goes from 0^n to $0v$ by passing through all vertices of the form $0x$ (this is simply a copy of the Hamiltonian path in dimension $(n-1)$, minus the edge from v to 0^{n-1}), then an edge from $0v$ to $1v$, then a path from $1v$ to 10^{n-1} that passes through all vertices of the form $1x$, and finally an edge from 10^{n-1} to 0^n. This completes the proof of the Theorem.

When we start from 0^n and we follow the Hamiltonian tour described in the above proof, we find an ordering of all the n-bit binary strings such that each string in the sequence differs from the previous string in only one bit. Such an ordering is called a Gray code (from the name of the inventor) and it has various application.