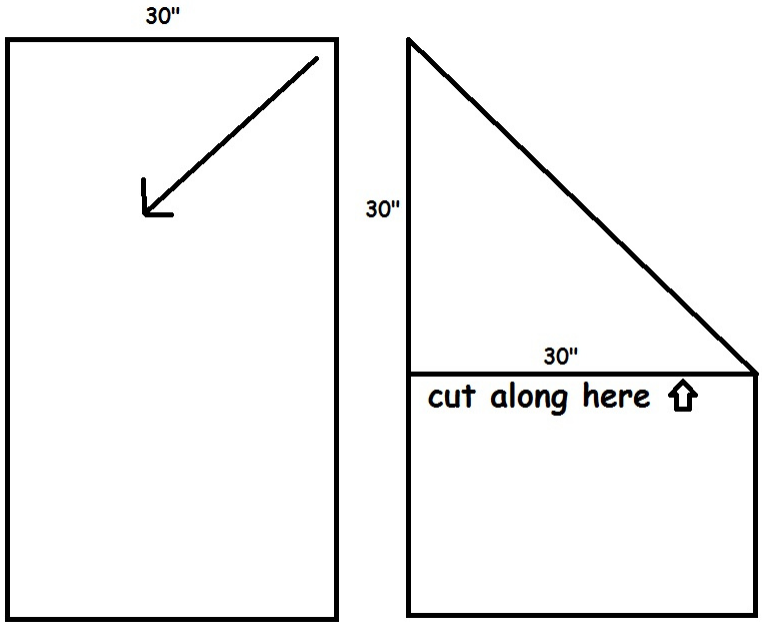
CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 5W

1. Paper GCD

Given a sheet of paper such as this one, and no rulers, describe a method to find the GCD of the width and the height of the paper. You can fold or tear the paper however you want, and ultimately you should produce a square piece whose side lengths are equal to the GCD.

Answer: We can fold the smaller side diagonally onto the larger side, and tear the paper from where the fold lands.



If we started with height and width equal to a and b, this gives us a piece of paper with side lengths $a-b$ and b (assuming that $a > b$). Note that if $a-b > b$, the next time we end up with side lengths $a-2b$ and b. So after a few steps we must reach $a \pmod b$ and b, at which we start subtracting from b.

Continuing this method is similar to the Euclidean algorithm and therefore results in reaching 0 at some point. Right before reaching 0, we must have a square piece of paper whose side lengths are the GCD.

2. Baby Fermat

Assume that a does have a multiplicative inverse $\pmod m$. Let us prove that its multiplicative inverse can be written as $a^k \pmod m$ for some $k \geq 0$.

- Consider the sequence $a, a^2, a^3, \ldots \pmod m$. Prove that this sequence has repetitions.

  Answer: There are only m possible values $\pmod m$, and so after the m-th term we should see repetitions.

- Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

  Answer: If we multiply both sides by $(a^*)^j$, where a^* is the multiplicative inverse, we get $a^{i-j} \equiv 1 \pmod{m}$.

- Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j?

  Answer: We can rewrite $a^{i-j} \equiv 1 \pmod{m}$ as $a^{i-j-1}a \equiv 1 \pmod{m}$. Therefore a^{i-j-1} is the multiplicative inverse of $a \pmod{m}$.

3. Extended Euclid

In this problem we will consider the extended Euclid's algorithm.

1. Note that x mod y, by definition, is always x minus a multiple of y. So, in the execution of Euclid's algorithm, each newly introduced value can always be expressed as a "combination" of the previous two, like so:
   $gcd(2328, 440)$
   $= gcd(440, 128)$ [$128 \equiv 2328 \bmod 440 \equiv 2328 - 5 \times 440$]
   $= gcd(128, 56)$ [$56 \equiv 440 \bmod 128 \equiv 440 - \_\_\_\_ \times 128$] **Answer:** 3
   $= gcd(56, 16)$ [$16 \equiv 128 \bmod 56 \equiv 128 - \_\_\_\_ \times 56$] **Answer:** 2
   $= gcd(16, 8)$ [$8 \equiv 56 \bmod 16 \equiv 56 - \_\_\_\_ \times 16$] **Answer:** 3
   $= gcd(8, 0)$ [$0 \equiv 16 \bmod 8 \equiv 16 - 2 \times 8$]
   $= 8$.
   (Fill in the blanks)

2. Now working back up from the bottom, we will express the final gcd above as a combination of the two arguments on each of the previous lines:
   8
   $= 1 \times 8 + 0 \times 0 = 1 \times 8 + (16 - 2 \times 8)$
   $= 1 \times 16 - 1 \times 8$
   $= \_\_\_\_ \times 56 + \_\_\_\_ \times 16$ [Hint: Remember, $8 = 56 - 3 \times 16$. Substitute this into the above line...]
   Answer: $1 \times 16 - 1 \times (56 - 3 \times 16) = -1 \times 56 + 4 \times 16$
   $= \_\_\_\_ \times 128 + \_\_\_\_ \times 56$ [Hint: Remember, $16 = 128 - 2 \times 56$]
   Answer: $4 \times 128 - 9 \times 56$
   $= \_\_\_\_ \times 440 + \_\_\_\_ \times 128$
   Answer: $-9 \times 440 + 31 \times 128$
   $= \_\_\_\_ \times 2328 + \_\_\_\_ \times 440$
   Answer: $31 \times 2328 - 164 \times 440$

3. In the same way as just illustrated in the previous two parts, calculate the gcd of 17 and 38, and determine how to express this as a "combination" of 17 and 38.

   Answer: $\gcd(17, 38) = 1 = 13 \times 38 - 29 \times 17$; also, more simply, $-4 \times 38 + 9 \times 17$, but the algorithm produces the former.

4. What does this imply, in this case, about the multiplicative inverse of 17, in arithmetic mod 38?

   Answer: It is equal to -29, which is equal to 9.

4. Product of Two

Suppose that $p > 2$ is a prime number and S is a set of numbers between 1 and $p-1$ such that $|S| > \frac{p}{2}$. Prove that any number $1 \leq x \leq p-1$ can be written as the product of two (not necessarily distinct) numbers in S, mod p.

Answer: Given x, consider the set T defined as $\{xy^{-1} \pmod{p} : y \in S\}$. Note that the set T has the same cardinality as S, because for $y_1 \neq y_2 \pmod{p}$, we have $xy_1^{-1} \neq xy_2^{-1} \pmod{p}$ (if not, we can multiply both sides by x^{-1}, and take the inverse to get a contradiction).

Therefore the set S and T must have a nonempty intersection. So there must be $y_1, y_2 \in S$ such that $xy_1^{-1} = y_2 \pmod{p}$. But this means that $x = y_1 y_2 \pmod{p}$.