CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Midterm 2

PRINT Your Name: ______________________________, ______________________________
(last) (first)

SIGN Your Name: ______________________________

PRINT Your Student ID: ______________________________

CIRCLE Your Exam Room: 2050 VLSB 10 EVANS OTHER

Name of the person sitting to your left: ______________________________

Name of the person sitting to your right: ______________________________

The discussion section you usually attend:

×	#	Time	TA
	9	10–11am	Caroline
	2	11–12pm	Favian
	3	12–1pm	Nathan Z.
	10	1–2pm	Diana

×	#	Time	TA
	4	4–5pm	Jerry
	5	4–5pm	Kunal
	7	4–5pm	Nathan M.

×	#	Time	TA
	1	5–6pm	Lexing
	6	5–6pm	Akshay
	8	5–6pm	Anna

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- You should not write anything, including your SID or name, after time is up.
- We will not grade anything on the extra pages (Page 12 and Pages 14–16) or outside of the space provided for a problem unless we are clearly told in the space provided for the question to look there.
- Except special notices, you must show your work to get credits.
- You may consult two single-sided sheets (or one double-sided sheet) of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- You have 110 minutes. There are 10 questions for a total of 100 points plus 15 bonus points. Use the number of points as a rough guide for the amount of time to allocate to that question.
- There are 16 pages (sides) on the exam. Notify a proctor immediately if a page is missing.

Do not turn this page until your instructor tells you to do so.

1. Probability + Stable Marriage (4 points)

Men	Preference Lists
1	C > A > B
2	B > C > A
3	B > C > A

Women	Preference Lists
A	2 > 1 > 3
B	3 > 1 > 2
C	3 > 2 > 1

Given the preference lists above, if we randomly and uniformly decide a pairing, what is the probability that the pairing is stable and optimal for men or women?

2. Probability + Polynomial (4 points)

$P(x)$ is a polynomial of degree at most 1, and each coefficient is decided randomly and uniformly from $\{0,1,2,3\}$. If we consider mod 4, observe $P(0) = 2$, and let random variable X be $P(2)$. What is the distribution of X?

3. Apples (8 points, 4 points for each part)

(a) You want to distribute 10 indistinguishable apples to Margo, Edith, Agnes, and Lucy. How many different ways are there? (You do not need to calculate the exact value of $\binom{a}{b}$ or $a!$.)

(b) Following Part (a), you also want to make sure Agnes gets at least 3 apples and Lucy gets at least 1 apple. How many different ways are there? (You do not need to calculate the exact value of $\binom{a}{b}$ or $a!$.)

4. Superheros and Gifts (10 points, 4/6 points for each part)

(a) Nick has 5 gifts (we do not know they are distinguishable or indistinguishable). For each gift, he randomly and uniformly gives it to one of the three superheroes: Hulk, Iron Man, and Captain America (each superhero has a $1/3$ probability to get the gift). However, Hulk will become angry if he does not get any gift. What is the probability that Hulk stays calm (not angry)? (Show your work and write down your answer as an integer divided by another integer.)

(They may be useful: $2^2 = 4, 2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$.)

(b) Following Part (a), Hulk will also become angry if he finds one or both of Iron Man and Captain America get more gifts than him. What is the probability that Hulk stays calm? (Show your work and write down your answer as an integer divided by another integer.)

(They may be useful: $\binom{2}{1} = 2$, $\binom{3}{1} = 3$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{5}{1} = 5$, $\binom{5}{2} = 10$.)

5. Traffic Dilemma (10 points, 6/4 points for each part)

(a) Your favorite team plays 50% games at Memorial Stadium, 40% games at AT&T Park, and 10% games at Coliseum. Its winning percentages are 80%, 50%, and 100% at Memorial Stadium, AT&T Park, and Coliseum, respectively. When driving, you hear news that the team just won a game. However, you forget where the game was, so you are not sure if you should detour to avoid traffic near Memorial Stadium. What is the probability that the game was at Memorial Stadium?

(b) Following Part (a), your friend reminds you that, although the driving time (from your current location to your destination) without detouring will be increased from 35 minutes to 70 minutes if there is a game at Memorial Stadium, the driving time (also from your current location to your destination) with detouring still needs 50 minutes. Given this information, should you detour? Justify your answer.

6. TRUE or FALSE — First Round (18 points, 8/10 points for each part)

For any of the following statements, claim TRUE or FALSE first. If you claim TRUE, prove it. If you claim FALSE, disprove it.

(a) You roll a fair dice twice. Let A be the event that you get the same outcome on both rolls and B be the event that you get "5" on your first roll.
Statement: A and B are dependent events.

TRUE FALSE

(b) Assume each regrade request is submitted and processed independently and the probability that a request is determined as an invalid one is $4/5$. Every student can have 3 non-penalized invalid requests (an invalid request means the corresponding number of points not changed), but the 4th one will result in 4-point penalty. You want to submit 4 requests where 1 request is for 5 points (the decision will be either 5 more points or no change) and each of the other 3 requests is for 1 point (the decision will be either 1 more point or no change).

Statement: submitting those 4 requests is advantageous.

TRUE FALSE

(They may be useful: $\left(\frac{4}{5}\right)^2 = 0.64$; $\left(\frac{4}{5}\right)^3 = 0.512$; $\left(\frac{4}{5}\right)^4 = 0.4096$; $\left(\frac{4}{5}\right)^5 = 0.32768$.)

7. Combinatorial Proof (10 points)

Use a combinatorial proof to prove the following statement:

$$2n\binom{3n-1}{n-1} = \sum_{i=1}^{n} \left(i\binom{n}{i}\binom{2n}{i} \right).$$

8. TRUE or FALSE — Second Round (24 points, 12 points for each part)

For any of the following statements, claim TRUE or FALSE first. If you claim TRUE, prove it. If you claim FALSE, disprove it (*e.g.*, provide a counterexample).

(a) There are 4 bins, and each bin has some black balls and some white balls. When you draw a ball from Bin i, the probability of getting a black ball is p_i, where $i = 1, 2, 3, 4$, and you are given that $p_1 \leq p_2$ and $p_3 \leq p_4$. Now, you put all balls in Bin 1 and Bin 3 to an empty bin, Bin 5, and you also put all balls in Bin 2 and Bin 4 to an empty bin, Bin 6. Define p_5, p_6 as the probabilities of getting a black ball from Bin 5 and Bin 6, respectively.

Statement: $p_5 \leq p_6$.

TRUE FALSE

(b) Statement: for any three events A_1, A_2, A_3 where $0 < \Pr[A_i] < 1$ for $i = 1, 2, 3$, if $\Pr[A_1 \cap A_2 \cap A_3] = \Pr[A_1] \times \Pr[A_2] \times \Pr[A_3]$, then A_1, A_2, A_3 are pairwise independent.

TRUE FALSE

9. Secret Sharing (12 points)

The United Nations consists of n countries ($n > 3$), and they are working with Thor and Loki. A vault can be opened with a secret s, and it should only be opened in any one of three situations:

1. All n countries in the UN help.
2. At least m countries ($0 < m < n-2$) get together with Thor.
3. At least $m+2$ countries ($0 < m < n-2$) get together with Loki.

Odin works over $GF(p)$ where p is sufficiently large and uses "only one" polynomial to implement a secret sharing scheme. In the scheme, $P(x)$ is with degree at most $n-1$, $P(0)$ is the secret, the i-th country gets a point $P(i)$, Thor gets $(n-m)$ points $P(n+1), P(n+2), \ldots, P(n+(n-m))$, and Loki gets $(n-m-2)$ points $P(n+(n-m)+1), P(n+(n-m)+2), \ldots, P(n+(n-m)+(n-m-2))$. Does this scheme work?

Hints: given $n > 3$ and $0 < m < n-2$, we suggest you follow one of the following three paths:

1. Always Works! Claim the scheme works for all n, m and justify your claim.
2. Sometimes Works! Claim the scheme works for some (but not all) n, m, provide the condition that it works, and justify your claim and condition.
3. Never Works! Claim the scheme does not work for all n, m and justify your claim.

(Extra Page: Remember to clearly tell in the space provided for a question to look here.)

10. Probability + Error Correction (15 bonus points)

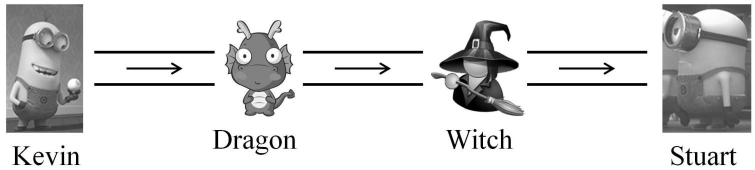

Figure 1: Minions using error correction codes [http://despicableme.wikia.com/wiki/Despicable_Me_Wiki].

Kevin wants to send a message of 10 ordered packets to Stuart. Each packet will first go through a hungry dragon. For each packet, there is a $1/10$ probability to be eaten by the dragon. Each remaining packet will then go through a witch. For each remaining packet, there is a $8/9$ probability to be changed by the witch. Given this scenario, how many packets should Kevin send so that the probability that Stuart can recover the message is maximized? Justify your answer (some approximation or estimation is encouraged for this question).

The following table of $\binom{n}{k}$ may be useful.

		n									
		10	11	12	13	14	15	16	17	18	19
k	0	1	1	1	1	1	1	1	1	1	1
	1	10	11	12	13	14	15	16	17	18	19
	2	45	55	66	78	91	105	120	136	153	171
	3	120	165	220	286	364	455	560	680	816	969
	4	210	330	495	715	1001	1365	1820	2380	3060	3876
	5	252	462	792	1287	2002	3003	4368	6188	8568	11628
	6	210	462	924	1716	3003	5005	8008	12376	18564	27132
	7	120	330	792	1716	3432	6435	11440	19448	31824	50388
	8	45	165	495	1287	3003	6435	12870	24310	43758	75582
	9	10	55	220	715	2002	5005	11440	24310	48620	92378
	10	1	11	66	286	1001	3003	8008	19448	43758	92378

		n									
		10	11	12	13	14	15	16	17	18	19
k	0	1	1	1	1	1	1	1	1	1	1
	1	10	11	12	13	14	15	16	17	18	19
	2	45	55	66	78	91	105	120	136	153	171
	3	120	165	220	286	364	455	560	680	816	969
	4	210	330	495	715	1001	1365	1820	2380	3060	3876
	5	252	462	792	1287	2002	3003	4368	6188	8568	11628
	6	210	462	924	1716	3003	5005	8008	12376	18564	27132
	7	120	330	792	1716	3432	6435	11440	19448	31824	50388
	8	45	165	495	1287	3003	6435	12870	24310	43758	75582
	9	10	55	220	715	2002	5005	11440	24310	48620	92378
	10	1	11	66	286	1001	3003	8008	19448	43758	92378

(Extra Page: Remember to clearly tell in the space provided for a question to look here.)

(Extra Page: Remember to clearly tell in the space provided for a question to look here.)