CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 4B

What is the **sample space** (*i.e.*, the experiment and its set of possible outcomes)?

What is the **probability** of each outcome (sample point)?

What is the **event** we are interested in (*i.e.*, which subset of the sample space)?

Compute the probability of the event by adding up the probabilities of the sample points inside it.

1. Balls from a Bag

There are 3 different balls (#1, #2, and #3) in a bag, and we want to select 2 balls from it.

(a) If order does matter with sampling with replacement,

(1) What is the sample space?
(2) What is the probability of each outcome?
(3) Which outcomes are in the event that we select Ball #2?
(4) What is the probability that we select Ball #2?

(b) If order does matter with sampling without replacement,

(1) What is the sample space?
(2) What is the probability of each outcome?
(3) Which outcomes are in the event that we select Ball #2?
(4) What is the probability that we select Ball #2?

(c) If order does not matter with sampling without replacement,

(1) What is the sample space?
(2) What is the probability of each outcome?
(3) Which outcomes are in the event that we select Ball #2?
(4) What is the probability that we select Ball #2?

(d) If order does not matter with sampling with replacement,

(1) What is the sample space?
(2) What is the probability of each outcome?
(3) Which outcomes are in the event that we select Ball #2?
(4) What is the probability that we select Ball #2?

(e) Are the probabilities equal to

$$\frac{\text{number of sample points in the event}}{\text{number of sample points in the sample space}}?$$

(f) Compare the probabilities with "order does matter" and "order does not matter" and explain what you observe.

Answer:

(a) (1) $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.
(2) $1/9$ for each outcome.
(3) $(1,2),(2,1),(2,2),(2,3),(3,2)$.
(4) $5/9$.

(b) (1) $\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$.
(2) $1/6$ for each outcome.
(3) $(1,2),(2,1),(2,3),(3,2)$.
(4) $4/6$.

(c) (1) $\{\{1,2\},\{1,3\},\{2,3\}\}$.
(2) $1/3$ for each outcome.
(3) $\{1,2\},\{2,3\}$.
(4) $2/3$.

(d) (1) $\{\{1,1\},\{1,2\},\{1,3\},\{2,2\},\{2,3\},\{3,3\}\}$.
(2) $1/9$ for $\{i,j\}$ where $i=j$ and $2/9$ for $\{i,j\}$ where $i \neq j$.
(3) $\{1,2\},\{2,2\},\{2,3\}$.
(4) $5/9$.

(e) Yes for (a), (b), and (c). No for (d).

(f) They are the same ($5/9$ with sampling with replacement and $2/3$ with sampling without replacement). This shows that "order does matter" and "order does not matter" do not matter. They can also be computed by $1-(\frac{2}{3})(\frac{2}{3})$ and $1-(\frac{2}{3})(\frac{1}{2})$, respectively.

2. Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5?

Answer:

(a) $\frac{18!5!}{22!} = \frac{1}{1463}$.

(b) $\left(\frac{2}{3}\right)^8 = \frac{256}{6561}$.

(c) $\left(\frac{3}{4}\right)^5 = \frac{243}{1024}$ (sampling with replacement) or $\frac{\binom{15}{5}}{\binom{20}{5}}$ (sampling without replacement).

3. Polynomial vs. Probability

$p(x)$ is a polynomial of degree 1, and both of its coefficients are decided randomly and uniformly.

(a) Consider mod 3 and $p(1)=2$. What are the probabilities of $p(0)=0,1,2$?

(b) Consider mod 3 and $p(2) = 2$. What are the probabilities of $p(0) = 0, 1, 2$?

(c) Consider mod 4 and $p(1) = 2$. What are the probabilities of $p(0) = 0, 1, 2, 3$?

(d) Consider mod 4 and $p(2) = 2$. What are the probabilities of $p(0) = 0, 1, 2, 3$?

Answer:

$p(x)$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2
x	0	1	2
$x+1$	1	2	0
$x+2$	2	0	1
$2x$	0	2	1
$2x+1$	1	0	2
$2x+2$	2	1	0

$p(x)$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
x	0	1	2	3
$x+1$	1	2	3	0
$x+2$	2	3	0	1
$x+3$	3	0	1	2
$2x$	0	2	0	2
$2x+1$	1	3	1	3
$2x+2$	2	0	2	0
$2x+3$	3	1	3	1
$3x$	0	3	2	1
$3x+1$	1	0	3	2
$3x+2$	2	1	0	3
$3x+3$	3	2	1	0

If $p(x)$ is a polynomial of degree "exactly" 1:

(a) $1/2, 1/2, 0$.

(b) $1/2, 1/2, 0$.

(c) $1/3, 1/3, 0, 1/3$.

(d) $2/3, 0, 1/3, 0$.

If $p(x)$ is a polynomial of degree "at most" 1:

(a) All are $1/3$.

(b) All are $1/3$.

(c) All are $1/4$.

(d) $1/2, 0, 1/2, 0$.