CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 6A

1. Variance

Prove or give a counterexample: for any random variables X and Y, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

Let X be any random variable such that $\text{Var}(X) \neq 0$. Then, if we set $Y = X$, $\text{Var}(X+Y) = \text{Var}(2X) = \mathbb{E}((2X)^2) - (\mathbb{E}(2X))^2 = 4(\mathbb{E}(X^2) - \mathbb{E}(X)^2) = 4\text{Var}(X) \neq \text{Var}(X) + \text{Var}(Y)$.

2. Cookies

TAs are giving out cookies to students in section! To make it more interesting, we give exactly 10 cookies to a student with probability $\frac{1}{2}$, otherwise we give either 1, 2, or 3 cookies randomly with equal probability. What is the expected number of cookies you'd get? What is its variance?

$$\begin{aligned}
\mathbb{E}(X) &= \frac{1}{2}\cdot 10 + \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 \\
&= 5 + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} \\
&= 6. \\
\text{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= \left(\frac{1}{2}\cdot 10^2 + \frac{1}{6}\cdot 1^2 + \frac{1}{6}\cdot 2^2 + \frac{1}{6}\cdot 3^2\right) - (6)^2 \\
&= \frac{157}{3} - 36 \\
&= \frac{49}{3}.
\end{aligned}$$

3. Bernoulli and Binomial Distribution

A random variable X is called a Bernoulli random variable with parameter p if $X = 1$ with probability p and $X = 0$ with probability $1-p$.

(a) Calculate $\mathbb{E}(X)$ and $\text{Var}(X)$.

$\mathbb{E}(X) = p\cdot 1 + (1-p)\cdot 0 = p$;

$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = p\cdot 1^2 + (1-p)\cdot 0^2 - p^2 = p(1-p)$.

(b) A Binomial random variable with parameters n and p is defined to be the sum of n independent, identically distributed Bernoulli random variables with parameter p. If Z is a Binomial random variable with parameters n and p, what are $\mathbb{E}(Z)$ and $\text{Var}(Z)$?

Let X_i be i.i.d. Bernoulli random variables with parameter p. By linearity of expectation, $\mathbb{E}(Z) = \sum_{i=1}^n \mathbb{E}(X_i) = np$. Moreover, since X_i's are independent, $\text{Var}(Z) = \sum_{i=1}^n \text{Var}(X_i) = np(1-p)$.

4. Will I Get My Package?

A deceitful delivery dude is out transporting n packages to n customers. Not only does he hand a random package to each customer, but he also opens a package before delivering with probability $\frac{1}{2}$ (independently of the choice of the package). Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}(X)$.

Define $X_i = \begin{cases} 1 & \text{if the } i\text{-th customer gets his/her own package unopened;} \\ 0 & \text{otherwise.} \end{cases}$

$\mathbb{E}(X) = \mathbb{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}(X_i)$.

$\mathbb{E}(X_i) = \Pr[X_i = 1] = \frac{1}{2n}$ since the i-th customer will get his/her package with probability $\frac{1}{n}$ and it is unopened with probability $\frac{1}{2}$ and packages are opened independently.

Hence $\mathbb{E}(X) = n \cdot \frac{1}{2n} = \frac{1}{2}$. By linearity: $\mathbb{E}(X) = \mathbb{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}(X_i) = n \cdot \frac{1}{2n} = \frac{1}{2}$.

(b) Calculate the probability that two particular customers i, j receive their own packages unopened.

$\Pr[\text{both customers receive own packages}] = \frac{(n-2)!}{n!}$.

$\Pr[i = \text{unopened}, j = \text{unopened}] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Since the two events are independent, $\Pr[X_i = 1, X_j = 1] = \frac{1}{4n(n-1)}$.

(c) Compute $\text{Var}(X)$.

$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

$\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + \ldots + X_n)^2) = \mathbb{E}(\sum_{i,j} X_i X_j) = \sum_{i,j} \mathbb{E}(X_i X_j)$, where the last equality follows from using linearity of expectation.

If $i = j$ then $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i^2) = \frac{1}{2n}$; if $i \neq j$, then $\mathbb{E}(X_i X_j) = \Pr[X_i X_j = 1] \cdot 1 + \Pr[X_i X_j = 0] \cdot 0 = \frac{1}{4n(n-1)}$.

Hence, $\mathbb{E}(X^2) = \sum_{i,j} \mathbb{E}(X_i X_j) = \sum_i \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j) = n \cdot \frac{1}{2n} + n(n-1) \cdot \frac{1}{4n(n-1)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Thus, $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.