CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin Discussion 7D

1. Counting Edge

Consider an undirected graph G with n nodes, no multi-edges, and self-loops allowed.

(a) What is the maximum number of edges G can have?

(b) How many different G graphs can there be? Graphs with the same number of nodes are only different if they have different sets of edges (assume nodes are labeled).

(c) Consider a directed graph H with n nodes, multi-edges not allowed (edges (a,b), and (b,a) are considered different), and self-loops not allowed. Edge (c,d) denotes that node c knows node d (similarly, edge (d,c) denotes node d knows node c). What is the maximum number of edges H can have?

(d) Assume that any given edge exists randomly and independently with probability p. Pick any node u. What is the probability all other nodes know u, but u does not know any other nodes?

(e) If the same node u from the previous part does not know any other nodes, but all other nodes know u, can such a graph have a Eulerian tour?

(a) $\binom{n}{2}+n$.

(b) $2^{\binom{n}{2}+n}$.

(c) $2\binom{n}{2}$.

(d) $p^{n-1}(1-p)^{n-1}$.

(e) No. In-degree of u is not equal to the out-degree of u.

2. Number of Heads Revisited

Consider an undirected graph G with n nodes, no multi-edges, and no self-loops. Consider all possible edges between any two nodes in G, and let e be the number of total possible edges. Suppose the probability, p, that a given edge exists between any two nodes is independent. (Answer in terms of e).

(a) What is the probability that exactly 34 edges exist in this graph? (Assume $e \geq 34$).

This problem models the binomial distribution. Let X be the number of edges in the graph. Therefore, $\Pr[X=34] = \binom{e}{34}(p^{34})(1-p)^{e-34}$.

(b) What is the expected number of edges the graph will have? What is the variance?

$\mathbb{E}(X) = ep$. $\mathrm{Var}(X) = ep(1-p)$.

(c) Consider an undirected graph with no self-loops, no multi-edges, and with four nodes a,b,c,d, and probability that an edge exists between any two nodes is still p. What is the probability that the graph is connected?

The graph is never connected when there are 2 or 1 or 0 edges. The graph is sometimes connected when there are 3 edges (there are only four ways it cannot be connected, e.g. triangle and dot). In all other cases, the graph must be connected. The maximum number of edges the graph can have is 6. Therefore, $\Pr[\text{connected}] = 1-\left(\binom{6}{2}(p^2)(1-p)^{6-2}+\binom{6}{1}(p^1)(1-p)^{6-1}+\binom{6}{0}(p^0)(1-p)^{6-0}+4(p^3)(1-p)^{6-3}\right)$.

3. Introduction to Trees

Recall that a tree is a connected graph with no cycles, (and so no self-loops, and no multi-edges). Show that any tree with at least 2 nodes must have a node of degree 1.

Use contradiction.

Case 1: All nodes have degree 0 or degree ≥ 2. The nodes which have degree 0 are not connected to the graph, so the graph cannot be a tree (contradiction).

Case 2: All nodes have degree ≥ 2. Pick any node, and pick an edge connected to that node and traverse the graph. All nodes have degree ≥ 2, so every time you enter a node, you also leave it, so there will always be another edge to cover. Eventually, you will have to revisit a node, which means there is a cycle in the graph. Then, the graph cannot be a tree (contradiction).

4. Graph Gardening

Prove that if graph G is a tree with e edges and n nodes, then $e = n - 1$. Use induction on n.

Solution uses the fact that any tree with at least two nodes must have a node of degree 1. If you remove an arbitrary edge in the induction step, the solution needs to use strong induction.

Base Case: $n = 1$. One node, no possible edges, so $e = 0 = n - 1$.

Induction Hypothesis: Assume that any tree G with k nodes nodes has $k - 1$ edges.

Induction Step: Start with tree G which has $k + 1$ nodes. Identify the node u in G of degree 1 (must exist, by above fact). Remove the edge connecting u to the rest of the graph. The result is the disconnected node u and a graph G' with k nodes. Since G was a connected and had no cycles and G' was contained in G, G' is connected and has no cycles as well (any path between two nodes in G' does not use the edge which was removed). Using the induction hypothesis, G' has $k - 1$ edges. Reconnecting u to the graph, we add 1 edge. Therefore, the number of edges in G is $k + 1 - 1 = k$.